Exponentiation Operators for Asymmetric Interval Numbers and Their Properties

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Abstract. Asymmetric Interval Numbers (AINs) represent uncertain quantities by an interval and a representative value defined as the expectation with respect to the auxiliary distribution. Unlike classical interval numbers, which specify only admissible ranges, AINs additionally encode the directional character of uncertainty. To support arithmetic operations, each AIN is associated with a canonical piecewise-constant auxiliary distribution consisting of two uniform segments determined by the interval bounds and the representative value. This distribution serves as a computational tool for evaluating operations on uncertain quantities. The existing AIN arithmetic covers basic algebraic operations but does not support nonlinear transformations in which uncertainty appears in the exponent. This paper extends AIN arithmetic by deriving analytic exponentiation operators for scalar-base exponentiation k^X and exponentiation between two uncertain quantities X^Y. For k^X, the operator is obtained in closed form by applying the Law of the Unconscious Statistician (LOTUS) to the auxiliary distribution. For X^Y, the representative value is defined by a two-step LOTUS construction that evaluates the joint expectation under the product of auxiliary densities using numerical quadrature. Numerical experiments confirm consistency of both operators with Monte Carlo integration of the auxiliary densities. The proposed extension enables direct application of AINs in nonlinear decision and predictive models involving exponential-type relationships.

Keywords: Asymmetric Interval Numbers· AIN · Interval arithmetic.

1 Introduction

Decision-making and predictive modeling frequently rely on numerical values derived from measurements, expert assessments, or incomplete data [8]. Such values rarely represent exact quantities; they inherently contain uncertainty resulting from observation errors, limited knowledge, or variability of the analyzed phenomena. Representing these quantities as single deterministic numbers implicitly

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eliminates the associated uncertainty. Moreover, input variables are rarely used in their original form. In most practical models, they undergo nonlinear transformations describing preferences or system response, such as utility functions or exponential mappings. When uncertain quantities are replaced by deterministic values, these transformations operate only on a central estimate, ignoring the associated variability. Consequently, uncertainty is not confined to the input data itself but should also be properly propagated through the nonlinear transformations applied to it [1, 7].

A wide range of approaches has been proposed to model uncertain quantities, including interval arithmetic [3], fuzzy numbers [5], and probabilistic representations [15]. These methods provide well-established mechanisms for handling basic algebraic operations. However, when nonlinear transformations are applied, operations such as exponentiation or general functional mappings may require α-cut approximations, sampling procedures, or defuzzification techniques, which either substantially increase computational cost, change the representation form, or lead to information loss [10, 11].

Asymmetric Interval Numbers (AINs) were recently introduced [13] as a compact representation of uncertain quantities defined by a lower bound, an upper bound, and a representative value located inside the interval. Unlike the interval midpoint, the representative value is not constrained to the center of the interval and may reflect directional character of the uncertainty, such as a tendency toward one of the bounds [14]. AIN does not require knowledge of the underlying probability distribution of the modeled quantity. In practice, this distribution may be unknown, empirical, or difficult to estimate from limited information. Therefore, the arithmetic of AIN employs an auxiliary piecewise-uniform distribution defined on the subintervals determined by the interval bounds and the representative value. This construction does not imply that the actual distribution is uniform; rather, it serves as a computational tool for deriving the results of arithmetic operations, consistent with the available information [12, 13].

Consequently, nonlinear transformations of an AIN can be evaluated by applying the Law of the Unconscious Statistician (LOTUS) to the auxiliary distribution [2], used purely as a derivation device for obtaining the representative value of the transformed quantity. This provides the theoretical basis for defining nonlinear operators for Asymmetric Interval Numbers [13]. Nevertheless, a limitation remains. Operators for expressions in which uncertainty appears in the exponent or in both operands, such as k^X and X^Y where X and Y are AINs, are currently unavailable. This limitation is significant because many practical models rely on exponential-type relationships, including utility functions and predictive systems. Without a treatment of these expressions, uncertainty cannot be propagated through such models using AINs.

To address this issue, this work introduces exponentiation operators for Asymmetric Interval Numbers covering both a real-valued base raised to an AIN exponent and exponentiation between two AINs. For the scalar-base case k^X, the operator is obtained in closed form by applying LOTUS to the auxiliary distribution. For the two-operand case X^Y, the representative value is defined by

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a two-step LOTUS construction that evaluates the joint expectation under the product of auxiliary densities. In both cases, the results follow from the adopted construction rather than from heuristic transformation of interval bounds. As a result, nonlinear transformations can be performed while maintaining real-valued outcomes and interpretability.

The contributions of the paper are guided by the following research questions:

RQ1. Can the exponentiation of AINs be defined for expressions of the form k^X and X^Y, where X and Y are AINs?
RQ2. Does the resulting operator preserve the structural properties of the AIN representation under nonlinear transformation?
RQ3. How do the proposed operators compare with sampling-based uncertainty propagation in terms of accuracy and computational cost?

By addressing these questions, the paper extends AIN arithmetic to cover nonlinear exponential-type transformations, enabling direct application of AINs in decision and predictive models without requiring explicit distributional assumptions or simulation. This provides a practical mechanism for propagating uncertainty through nonlinear models while preserving the simplicity and interpretability of the AIN representation.

The remainder of the paper is organized as follows. Section 2 introduces the AIN representation and its auxiliary distribution. Section 3 presents the proposed definition of exponentiation for AINs. Section 4 provides analytical properties and illustrative examples. Section 5 compares the proposed approach with sampling-based propagation. Section 6 concludes the paper.

2 Preliminaries

This section briefly recalls the fundamental properties of Asymmetric Interval Numbers required for the development of the exponentiation operator. The presentation follows the original formulation [13] and is restricted to results necessary for nonlinear transformations.

An Asymmetric Interval Number is defined as a triple

$$X = [a, b]_c,$$

where $a, b \in \mathbb{R}$, $a \leq b$, and $c \in (a, b)$ is a distinguished value internal to the interval, referred to as the representative value. The value c is not constrained to the midpoint of the interval; its position within $[a, b]$ reflects the asymmetry of the uncertainty, characterized by the distances $c - a$ and $b - c$ from the representative value to the respective bounds.

To define arithmetic operations, each AIN is equipped with an auxiliary piecewise-uniform density on $[a, b]$:

$$p_X(x) = \begin{cases} \alpha, & a \leq x < c, \\ \beta, & c \leq x \leq b, \\ 0, & \text{otherwise}, \end{cases}$$

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where the parameters α and β are determined by two conditions: the normalization constraint

$$\int_a^b p_X(x)\,dx = 1$$

and the requirement that the expected value with respect to p_X coincides with the representative value

$$E(X) = \int_a^b x\,p_X(x)\,dx = c,$$

which yields

$$\alpha = \frac{b-c}{(b-a)(c-a)}, \qquad \beta = \frac{c-a}{(b-a)(b-c)}.$$

The above expressions require $c \in (a,b)$. The degenerate case $a = b$ corresponds to a crisp number with no uncertainty. The boundary cases $c = a$ or $c = b$ are excluded from the standard AIN definition, as they lead to a density concentrated entirely on one side of the interval.

This construction does not model the unknown distribution of the represented quantity. It provides a canonical computational device that enables evaluation of arithmetic operations on AINs.

For a measurable function $g : \mathbb{R} \to \mathbb{R}$, the representative value of the transformed AIN is obtained by applying the Law of the Unconscious Statistician to the auxiliary density:

$$E[g(X)] = \int_a^b g(x)\,p_X(x)\,dx.$$

If g is monotonic on $[a,b]$, the transformed AIN takes the form

$$g(X) = [\min(g(a), g(b)),\ \max(g(a), g(b))]_{E[g(X)]}.$$

The auxiliary distribution thus provides a systematic mechanism for defining nonlinear operators on AINs: the interval bounds are obtained from the range of g over $[a,b]$, while the representative value of the result is derived from the auxiliary construction. Formal properties of the resulting operators, including consistency with crisp arithmetic and location of the representative value within the transformed bounds, are established in Section 4.

3 Exponentiation of Asymmetric Interval Numbers

The preliminaries established that nonlinear transformations of a single AIN can be defined by applying LOTUS [2] to the auxiliary density: the interval bounds are obtained from the range of the transformation, and the representative value is derived from the auxiliary construction. We now use this mechanism to define exponentiation operators.

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Two cases are considered:

$$k^X \quad \text{and} \quad X^Y,$$

where X and Y are AINs and $k > 0$ is a scalar. The first case involves a single AIN and admits a direct application of LOTUS. The second case involves two AINs and requires a compositional construction, which we define as a sequence of one-dimensional AIN operations.

3.1 Exponentiation with a scalar base

Let $X = [a, b]_c$ be an AIN and let p_X denote its auxiliary density with parameters α and β. For $k > 0$, $k \neq 1$, the function $g(x) = k^x$ is a monotonic transformation of a single variable. The representative value of the transformed AIN is therefore obtained directly by applying LOTUS:

$$E(k^X) = \int_a^b k^x \, p_X(x) \, dx.$$

Because the density is piecewise constant, the integral splits into two parts:

$$E(k^X) = \alpha \int_a^c k^x \, dx + \beta \int_c^b k^x \, dx.$$

Using the identity $\int k^x \, dx = k^x / \ln k$, we obtain a closed-form expression.

Lemma 1. *Let $X = [a, b]_c$ be an AIN and let $k > 0$, $k \neq 1$. Then*

$$E(k^X) = \frac{\alpha(k^c - k^a) + \beta(k^b - k^c)}{\ln k}.$$

Proof. Substituting the auxiliary density into the LOTUS integral and evaluating gives

$$\begin{aligned} E(k^X) &= \alpha \int_a^c k^x \, dx + \beta \int_c^b k^x \, dx \\ &= \alpha \, \frac{k^c - k^a}{\ln k} + \beta \, \frac{k^b - k^c}{\ln k} \\ &= \frac{\alpha(k^c - k^a) + \beta(k^b - k^c)}{\ln k}, \end{aligned}$$

which proves the claim. □

Proposition 1 (Scalar-base exponentiation). *Let $X = [a, b]_c$ be an AIN and let $k > 0$, $k \neq 1$. The exponentiation of k by X is defined as*

$$k^X = [\min(k^a, k^b), \ \max(k^a, k^b)]_{E(k^X)},$$

where the representative value is given by Lemma 1.

The bounds follow from the monotonicity of $x \mapsto k^x$ (increasing for $k > 1$, decreasing for $0 < k < 1$), while the representative value is obtained in closed form from the auxiliary density.

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3.2 Exponentiation of two AINs

We now consider the case $Z = X^Y$, where both the base and the exponent are AINs. Let

$$X = [a_X, b_X]_{c_X}, \quad Y = [a_Y, b_Y]_{c_Y}, \quad Z = [a_Z, b_Z]_{c_Z},$$

with $a_1 > 0$.

In contrast to the scalar-base case, the AIN framework does not define a joint auxiliary representation for a pair of quantities. To evaluate expressions involving two AINs, the construction is extended to the product space by combining the auxiliary densities of the two quantities. This extension does not introduce any independence assumption; it provides a canonical lifting of the AIN representation that adds no information beyond that contained in the individual operands. Under this assumption, the representative value is given by the double integral

$$c_Z = \iint_{[a_X,b_X]\times[a_Y,b_Y]} x^y \, p_X(x) \, p_Y(y) \, dx \, dy.$$

Since $x^y = e^{y \ln x}$ and $x \in [a_X, b_X]$ with $a_X > 0$, the function x^y is continuous on the compact rectangle $[a_X, b_X] \times [a_Y, b_Y]$ and therefore Lebesgue integrable. As the integrand is nonnegative, Tonelli's theorem [4] allows the double integral to be written as Lemma 1:

$$\mathcal{L}_Y[x^y] = \int_{a_Y}^{b_Y} x^y \, p_Y(y) \, dy = \frac{\alpha_Y(x^{c_Y} - x^{a_Y}) + \beta_Y(x^{b_Y} - x^{c_Y})}{\ln x}.$$

Applying the representative-value functional with respect to p_X then gives:

$$c_Z = \int_{a_X}^{b_X} \mathcal{L}_Y[x^y] \, p_X(x) \, dx = \alpha_X \int_{a_X}^{c_X} \mathcal{L}_Y[x^y] \, dx + \beta_X \int_{c_X}^{b_X} \mathcal{L}_Y[x^y] \, dx.$$

No closed form in elementary functions is available for the outer integral, as the antiderivative of $\frac{x^u}{\ln x}$ with respect to x is not expressible in elementary functions; the integral is therefore evaluated numerically.

Since x^y is continuous on the compact rectangle $[a_X, b_X] \times [a_Y, b_Y]$ with $a_X > 0$, the extrema exist by the Weierstrass theorem [9]. The interval bounds of X^Y are defined as:

$$a_Z = \min\{x^y : x \in [a_X, b_X], \, y \in [a_Y, b_Y]\},$$
$$b_Z = \max\{x^y : x \in [a_X, b_X], \, y \in [a_Y, b_Y]\}.$$

Lemma 2 (Corner attainment of bounds). *Let $a_X > 0$. Then*

$$a_Z = \min\{a_X^{a_Y}, \, a_X^{b_Y}, \, b_X^{a_Y}, \, b_X^{b_Y}\}, \qquad b_Z = \max\{a_X^{a_Y}, \, a_X^{b_Y}, \, b_X^{a_Y}, \, b_X^{b_Y}\}.$$

Proof. We first exclude interior critical points. For $f(x, y) = x^y = e^{y \ln x}$,

$$\partial_x f = y x^{y-1}, \qquad \partial_y f = x^y \ln x.$$

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Both vanish simultaneously only at $(x, y) = (1, 0)$, where $f(1, 0) = 1$. This value is neither a global minimum nor a global maximum on the rectangle unless f is constant there (which can happen only in degenerate cases such as $a_X = b_X = 1$ or $a_Y = b_Y = 0$). Hence any global extremum must lie on the boundary of $[a_X, b_X] \times [a_Y, b_Y]$.

The boundary consists of four edges. On the left and right edges $x \in \{a_X, b_X\}$, the function $y \mapsto x^y = e^{y \ln x}$ has derivative $x^y \ln x$, which has constant sign on the entire interval $[a_Y, b_Y]$ for fixed $x \neq 1$, and therefore $y \mapsto x^y$ is monotone on this interval; for $x = 1$ the function equals the constant 1. In both cases the extrema over y are attained at $y \in \{a_Y, b_Y\}$. On the top and bottom edges $y \in \{a_Y, b_Y\}$, the function $x \mapsto x^y$ has derivative yx^{y-1}, which has constant sign on $[a_X, b_X]$ for fixed $y \neq 0$, hence $x \mapsto x^y$ is monotone on this interval; for $y = 0$ the function equals the constant 1. In both cases the extrema over x are attained at $x \in \{a_X, b_X\}$. Hence all global extrema are attained at the four corners $\{a_X, b_X\} \times \{a_Y, b_Y\}$, which completes the proof.

Definition 1 (Exponentiation of AINs). *Let $X = [a_X, b_X]_{c_X}$ and $Y = [a_Y, b_Y]_{c_Y}$ be AINs with $a_1 > 0$. The exponentiation of AINs is defined as*

$$Z = X^Y = [a_Z,\, b_Z]_{c_Z},$$

where a_Z and b_Z are defined above, and the representative value is

$$c_Z = \alpha_X \int_{a_X}^{c_X} \frac{\alpha_Y(x^{c_Y} - x^{a_Y}) + \beta_Y(x^{b_Y} - x^{c_Y})}{\ln x}\, dx + \beta_X \int_{c_X}^{b_X} \frac{\alpha_Y(x^{c_Y} - x^{a_Y}) + \beta_Y(x^{b_Y} - x^{c_Y})}{\ln x}\, dx.$$

Remark 1 (Numerical evaluation of representative value). Since $\mathcal{L}_Y[x^y]$ is smooth on $(a_X, b_X]$ for $a_X > 0$ and extends continuously to $x = 1$, the representative value c_Z can be evaluated numerically using standard quadrature rules.

Remark 2 (Removable Singularity of the Integrand at $a_X = 1$). When $a_X = 1$, the integrand in the representative value formula contains $\ln x$ in the denominator:

$$\frac{\alpha_Y(x^{c_Y} - x^{a_Y}) + \beta_Y(x^{b_Y} - x^{c_Y})}{\ln x}.$$

Since $\ln 1 = 0$, the point $x = 1$ is an apparent singularity of the integrand. To show that it is removable, we use the Taylor expansion $x^u = e^{u \ln x} = 1 + u \ln x + O(\ln^2 x)$ as $x \to 1$, which gives

$$\frac{x^u - x^v}{\ln x} = \frac{(u - v) \ln x + O(\ln^2 x)}{\ln x} \xrightarrow{x \to 1} u - v.$$

Equivalently, this expression admits the integral representation

$$\frac{x^u - x^v}{\ln x} = \int_v^u x^t\, dt,$$

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which confirms both the existence of the limit as $x \to 1$ and the numerical stability of the limiting value $u - v$. Applying this limit to each term yields

$$\lim_{x \to 1} \frac{\alpha_Y(x^{c_Y} - x^{a_Y}) + \beta_Y(x^{b_Y} - x^{c_Y})}{\ln x} = \alpha_Y(c_Y - a_Y) + \beta_Y(b_Y - c_Y).$$

Since the limit is finite, the integrand admits a continuous extension to $x = 1$; after such an extension the function is bounded in a neighbourhood of that point and hence Riemann integrable on the entire interval $[a_X, b_X]$.

From a numerical standpoint, direct evaluation of $\frac{x^u - x^v}{\ln x}$ near $x = 1$ leads to catastrophic cancellation. In practice, whenever $|x - 1| < \varepsilon$ for a chosen tolerance $\varepsilon > 0$, the integrand should be replaced by its limiting value $\alpha_Y(c_Y - a_Y) + \beta_Y(b_Y - c_Y)$.

Remark 3 (Compositional operator as algebraic alternative). An alternative construction defines X^Y via the pointwise identity $x^y = \exp(y \ln x)$ as a composition of existing AIN operations:

$$X^Y_{\text{comp}} = \exp\bigl(Y \cdot \ln X\bigr),$$

where each step is evaluated sequentially via LOTUS applied to the respective auxiliary density. This operator is fully closed-form and evaluates in constant time. Its representative value may differ from c_Z, since sequential application of nonlinear transformations does not in general commute with expectation. In our numerical experiments the compositional operator consistently produced larger representative values than c_Z; whether this reflects a systematic bias depends on the configuration of X and Y and we do not claim it holds in general. The compositional operator may be preferred in applications requiring constant-time evaluation.

3.3 Scalar-base exponentiation k^X

Remark 4 (Degenerate case). The auxiliary density construction requires $a < c < b$ and is therefore undefined for a degenerate AIN $X = [x, x]_x$. The operator k^X is extended to this case by the convention

$$k^{[x,x]_x} = [k^x, k^x]_{k^x},$$

consistent with standard real arithmetic. This extension is also compatible with the limiting behavior of the general formula as $a, b \to x$.

Proposition 2 (Interval bounds). *Let $X = [a, b]_c$ be an AIN and let $k > 0$, $k \neq 1$. Then*

$$k^X = [\min(k^a, k^b),\ \max(k^a, k^b)]_{E(k^X)}.$$

Proof. The function $x \mapsto k^x$ is strictly increasing for $k > 1$ and strictly decreasing for $0 < k < 1$. It therefore maps $[a, b]$ onto $[k^a, k^b]$ or $[k^b, k^a]$, respectively. □

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Proposition 3 (Representative value lies within bounds). *Let $X = [a, b]_c$ be an AIN and let $k > 0$, $k \neq 1$. Then*

$$E(k^X) \in [\min(k^a, k^b),\ \max(k^a, k^b)].$$

Proof. The expectation of a bounded function lies in the convex hull of its range. Since $x \mapsto k^x$ is monotonic on $[a, b]$, its range is the interval $[\min(k^a, k^b), \max(k^a, k^b)]$, and the claim follows. □

Proposition 4 (Jensen-type inequality). *Let $X = [a, b]_c$ be an AIN and let $k > 0$, $k \neq 1$. Then*

$$E(k^X) \geq k^c.$$

Proof. The function $x \mapsto k^x$ is convex for all $k > 0$, $k \neq 1$, since $\frac{d^2}{dx^2}k^x = (\ln k)^2 k^x > 0$. By construction of the auxiliary density, $E(X) = c$. Jensen's inequality applied to the probability measure p_X and the convex function $x \mapsto k^x$ yields $E(k^X) \geq k^{E(X)} = k^c$. □

Proposition 5 (Asymmetry coefficient after scalar-base exponentiation). *Let $X = [a, b]_c$ be an AIN and let $k > 0$, $k \neq 1$. Define $Z = k^X$ with*

$$Z = [a_Z, b_Z]_{c_Z}, \quad \textit{where} \quad a_Z = \min(k^a, k^b),\ b_Z = \max(k^a, k^b),\ c_Z = E(k^X).$$

Then the asymmetry coefficient of Z equals

$$A(Z) = \frac{a_Z + b_Z - 2c_Z}{b_Z - a_Z},$$

where c_Z is given by Lemma 1.

Proof. Substituting a_Z, b_Z, and c_Z into the definition of the AIN asymmetry coefficient $A = (a + b - 2c)/(b - a)$ yields the stated expression. The closed form for c_Z was established in Lemma 1. □

3.4 Exponentiation of two AINs X^Y

Proposition 6 (Domain condition and real-valuedness). *Let $X = [a_X, b_X]_{c_X}$ and $Y = [a_Y, b_Y]_{c_Y}$ be AINs with $a_X > 0$. Then $Z = X^Y$ as defined in Definition 1 produces a real-valued AIN with positive bounds and positive representative value.*

Proof. Since $a_X > 0$, we have $x > 0$ for all $x \in [a_X, b_X]$, hence $x^y > 0$ for all $y \in [a_Y, b_Y]$. Therefore $a_Z > 0$ and $b_Z > 0$.

For the representative value, write $\mathcal{L}_Y[x^y] = \int_{a_Y}^{b_Y} x^y\, p_Y(y)\, dy$. Since $x^y > 0$ for $x > 0$ and p_Y is a positive measure, the integrand is strictly positive, hence $\mathcal{L}_Y[x^y] > 0$ for all $x \in (a_X, b_X]$. The outer integral against the positive measure p_X is therefore strictly positive, giving $c_Z > 0$. □

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Proposition 7 (Consistency with scalar cases). *Let* $X = [a_X, b_X]_{c_X}$ *and* $Y = [a_Y, b_Y]_{c_Y}$ *be AINs with* $a_X > 0$*.*

1. *If* $Y = [n, n]_n$ *is degenerate, then* X^Y *coincides with scalar exponentiation* X^n*.*
2. *If* $X = [k, k]_k$ *is degenerate with* $k > 0$*,* $k \neq 1$*, then* $X^Y = k^Y$ *as defined in Proposition 1.*

Proof. For case 1, when Y is degenerate at n, the inner LOTUS transform reduces to evaluation at $y = n$:

$$\mathcal{L}_Y[x^y] = x^n.$$

The outer integral then gives $c_Z = \int_{a_X}^{b_X} x^n \, p_X(x) \, dx = E(X^n)$, and the bounds reduce to $\min(a_X^n, b_X^n)$ and $\max(a_X^n, b_X^n)$, recovering scalar exponentiation X^n.

For case 2, when X is degenerate at k, the outer integral reduces to evaluation at $x = k$:

$$c_{X^Y} = \mathcal{L}_Y[k^y] = \int_{a_Y}^{b_Y} k^y \, p_Y(y) \, dy = E(k^Y),$$

and the bounds reduce to $\min(k^{a_Y}, k^{b_Y})$ and $\max(k^{a_Y}, k^{b_Y})$, recovering k^Y. □

Remark 5. The logarithmic identity $\ln(X^Y) = Y \cdot \ln X$, which holds by construction for the compositional operator $X^Y_{\text{comp}} = \exp(Y \cdot \ln X)$ (Remark 3), does not hold in general for the two-step LOTUS operator defined in Definition 1, except in degenerate or special cases such as those covered by Proposition 7. The algebraic identity is a property of the compositional structure, not of the two-step LOTUS integral.

Remark 6. The properties established in this section confirm that the proposed operators are consistent extensions of the existing AIN arithmetic. The degenerate-case convention and Proposition 7 verify reduction to standard real arithmetic and to the scalar-base operator in all applicable special cases. Proposition 6 guarantees that the result of X^Y is a valid AIN with positive bounds and positive representative value whenever $a_X > 0$.

3.5 Illustrative examples

We present three examples illustrating the behavior of the proposed operators. Example 1 uses the closed-form expression of Lemma 1. Examples 2 and 3 use the two-step LOTUS construction of Definition 1, with the outer integral evaluated numerically.

Example 1: Asymmetry introduced by k^X*.* Let $X = [1, 5]_3$ be a symmetric AIN and let $k = 2$. The auxiliary density parameters are $\alpha = \frac{5-3}{(5-1)(3-1)} = \frac{1}{4}$ and $\beta = \frac{3-1}{(5-1)(5-3)} = \frac{1}{4}$. From Lemma 1:

$$c_Z = \frac{\frac{1}{4}(2^3 - 2^1) + \frac{1}{4}(2^5 - 2^3)}{\ln 2} = \frac{\frac{1}{4}(6) + \frac{1}{4}(24)}{\ln 2} = \frac{7.5}{0.6931} \approx 10.82,$$

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giving $k^X = [2, 32]_{10.82}$. The asymmetry coefficient of the result is

$$A(Z) = \frac{2 + 32 - 2 \cdot 10.82}{32 - 2} = \frac{12.36}{30} \approx 0.412.$$

Although the input AIN is symmetric ($c = 3$ is the midpoint of $[1, 5]$), the output exhibits positive asymmetry: the representative value $c_Z \approx 10.82$ lies closer to the left bound of $[2, 32]$. This is consistent with the Jensen-type inequality (Proposition 4): $c_Z \approx 10.82 > k^c = 8$, reflecting the convexity of $x \mapsto 2^x$.

Example 2: Naive exponentiation vs. the proposed operator. Consider $X = [2, 6]_3$ and $Y = [1, 3]_2$. A naive approach, applying exponentiation separately to bounds and representative values, produces

$$X^Y_{\text{naive}} = [2^1, 6^3]_{3^2} = [2, 216]_9.$$

The proposed operator (Definition 1) gives bounds $a_Z = 2^1 = 2$ and $b_Z = 6^3 = 216$ (corner case, since $a_1 = 2 \geq 1$), and the representative value is obtained from the two-step LOTUS construction:

$$c_Z = \alpha_X \int_2^3 \mathcal{L}_Y[x^y]\, dx + \beta_X \int_3^6 \mathcal{L}_Y[x^y]\, dx \approx 13.18,$$

giving $X^Y \approx [2, 216]_{13.18}$.

Both approaches produce the same interval bounds, but the representative values differ substantially: 9 (naive) vs. 13.18 (proposed). The naive value $c_X^{c_Y} = 3^2 = 9$ underestimates the representative value because it ignores the nonlinear amplification introduced by exponentiation over the full uncertainty range of both operands. The proposed operator accounts for this effect through the joint auxiliary density structure and agrees with Monte Carlo estimation to within sampling error.

Example 3: Asymmetry propagation through X^Y. Let $X = [2, 8]_6$ (negative asymmetry, $A(X) = -0.333$) and $Y = [1, 3]_{1.5}$ (positive asymmetry, $A(Y) = 0.5$). The proposed operator yields

$$X^Y \approx [2, 512]_{c_Z}, \quad c_Z \approx 28.79, \quad A(X^Y) \approx 0.895.$$

For comparison, using the same X with a symmetric exponent $Y' = [1, 3]_2$ ($A(Y') = 0$) gives

$$X^{Y'} \approx [2, 512]_{c_{Z'}}, \quad c_{Z'} \approx 66.65, \quad A(Z') \approx 0.746.$$

The interval bounds are identical in both cases, since they depend only on the endpoints. The representative values and asymmetry coefficients differ: Y has positive asymmetry, meaning its auxiliary mass is concentrated toward smaller values of y, which suppresses the representative value of X^Y relative to the symmetric case Y' and produces a higher asymmetry coefficient (0.895 vs. 0.746). This illustrates that the proposed operator is sensitive to the asymmetry of both operands, a property invisible to any method operating solely on interval endpoints.

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4 Numerical validation and computational efficiency

This section evaluates the proposed exponentiation operators by comparing the analytic representative values with reference values obtained from Monte Carlo (MC) sampling. For k^X, the comparison serves as a consistency check of the closed-form formula against its stochastic counterpart. For X^Y, it verifies that the two-step LOTUS construction agrees with MC numerical integration of the joint auxiliary density.

4.1 Experimental setup

For a given AIN $X = [a, b]_c$, samples were drawn from the auxiliary piecewise-uniform density p_X by selecting one of the two subintervals $[a, c)$ or $[c, b]$ with probabilities proportional to their normalized masses $\alpha(c - a)$ and $\beta(b - c)$, which sum to 1 by construction, and then drawing uniformly within the selected subinterval. For two AINs X and Y, the Monte Carlo estimate was computed as

$$\widehat{E}[g(X,Y)] = \frac{1}{N}\sum_{i=1}^{N} g(x_i, y_i),$$

where (x_i, y_i) are independent realizations drawn from the respective auxiliary densities, with N ranging from 10^3 to $3 \cdot 10^5$.

Monte Carlo sampling is used here only as a numerical integration technique applied to the auxiliary densities. All numerical experiments were implemented in Python (version 3.13) using the `asymintervals` library [12] for AIN arithmetic and NumPy [6] for random sampling. The two-step LOTUS integral for X^Y was evaluated using adaptive quadrature as implemented in `scipy.integrate.quad`.

Two cases were investigated. The first is scalar-base exponentiation k^X, for which Lemma 1 provides a closed-form expression. The second is exponentiation X^Y, for which the representative value is obtained from the two-step LOTUS construction in Definition 1.

4.2 Results for k^X

For the scalar-base case, the representative value is obtained from a single one-dimensional LOTUS integral evaluated in closed form (Lemma 1). The Monte Carlo estimate, computed by sampling from the same auxiliary density, provides a direct consistency check of the analytic formula. Table 1 presents results for selected AIN configurations and scalar bases.

Across all tested configurations, the relative difference between the analytic value and the Monte Carlo estimate remained on the order of 10^{-4}–10^{-3}, consistent with sampling variability at $N = 10^5$. This confirms that the closed-form expression correctly evaluates the LOTUS integral associated with the auxiliary density.

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Table 1. Scalar-base exponentiation k^X: analytic representative value vs. Monte Carlo estimate ($N = 10^5$).

a	b	c	k	Analytic	MC estimate
1	5	2	2	5.5303	5.5125
0	10	3	1.5	7.1051	7.0583
2	8	6	3	1824.1194	1817.8170

4.3 Results for X^Y

For the two-operand case, the representative value c_Z is defined by the two-step LOTUS construction (Definition 1), which evaluates the integral

$$c_Z = \int_{a_X}^{b_X} \int_{a_Y}^{b_Y} x^y \, p_X(x) \, p_Y(y) \, dy \, dx$$

using the analytically available inner integral $\mathcal{L}_Y[x^y]$ and adaptive quadrature for the outer integral. The Monte Carlo estimate of the same integral therefore provides a direct consistency check. Table 2 reports results for selected AIN configurations.

Table 2. Exponentiation X^Y: two-step LOTUS representative value vs. Monte Carlo estimate ($N = 10^5$).

a_X	b_X	c_X	a_Y	b_Y	c_Y	Two-step LOTUS	MC estimate
2	5	3	1	3	2	12.36	12.34
1	4	1.5	0.5	2	1	1.56	1.57
3	7	6	1	2	1.2	9.67	9.67

Across all tested configurations, the two-step LOTUS value agreed with the Monte Carlo estimate to within sampling variability, confirming that Definition 1 correctly evaluates the integral under the product of auxiliary densities. The residual discrepancy is consistent with the combined effect of MC sampling variability and quadrature tolerance.

4.4 Computational efficiency

The scalar-base operator k^X is evaluated using a closed-form expression involving a fixed computational cost independent of any sampling parameter. The two-step LOTUS operator for X^Y requires evaluation of a one-dimensional numerical integral; adaptive quadrature achieves the prescribed tolerance with substantially fewer function evaluations than Monte Carlo sampling, which requires $O(1/\varepsilon^2)$ samples to achieve root-mean-square error ε:

$$\mathrm{RMSE}_{\mathrm{MC}} = O\left(\frac{1}{\sqrt{N}}\right).$$

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Both proposed operators are therefore deterministic and reproducible, and substantially faster than sampling-based propagation at any prescribed accuracy level. This property is particularly relevant in decision and predictive models where uncertainty propagation must be performed repeatedly or in real time.

5 Conclusions

This paper extends the arithmetic of Asymmetric Interval Numbers by introducing exponentiation operators for expressions of the form k^X and X^Y. A closed-form operator was obtained for the scalar-base case k^X, while the two-operand case X^Y was defined through a two-step LOTUS construction based on the product of auxiliary densities. As a result, AIN arithmetic can now handle exponential-type nonlinear transformations under the domain condition $a_1 > 0$.

The proposed operators preserve the structure of the AIN representation. The resulting quantities remain real-valued AINs with positive bounds, the representative value retains its interpretation as an expectation with respect to the auxiliary distribution, and the operators reduce to standard arithmetic in degenerate cases. Numerical experiments confirmed agreement with Monte Carlo estimation within sampling variability, while the operators themselves remain deterministic and reproducible, with fixed computational cost.

The practical consequence of this extension is the ability to propagate uncertainty through models involving exponential relationships. This includes nonlinear utility functions, multiplicative production models such as Cobb–Douglas, and growth or decay processes. Previously, such models could be evaluated under uncertainty only via sampling-based simulation or by replacing uncertain inputs with point estimates. The proposed operators provide a deterministic alternative that preserves the structure of uncertainty throughout the computation.

Future work may consider extensions to additional classes of nonlinear transformations and the integration of the proposed arithmetic into decision-making and optimization frameworks. The presented results establish exponentiation as a well-defined operation in AIN arithmetic, enabling deterministic propagation of uncertainty in models involving exponential relationships.

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