Computing Wasserstein Distances Between Asymmetric Interval Numbers

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Abstract. Asymmetric interval numbers (AINs) extend classical intervals by incorporating an expected value. Each AIN is associated with a canonical probabilistic representation in the form of a piecewise-constant approximating distribution composed of two uniform distributions, uniquely determined so that the normalization condition holds, reflecting the distributional asymmetry. However, classical interval distances, such as the Hausdorff distance, depend only on interval endpoints and assign distance zero to AINs sharing the same support but differing in expected value. In this paper, we derive closed-form analytical formulas for the Wasserstein distances W_1, W_2, and W_∞ between AINs, exploiting the piecewise-linear structure of the quantile functions of the associated distributions. All three formulas are evaluated in constant time $\mathcal{O}(1)$, eliminating the need for numerical integration $\mathcal{O}(n)$. We prove that the proposed distances are proper metrics and establish their key structural properties, including translation invariance, positive homogeneity, continuity, and ordering $W_1 \leq W_2 \leq W_\infty$. We further demonstrate that these distances are sensitive to distributional asymmetry, making them strictly more informative than classical interval metrics.

Keywords: Asymmetric Interval Numbers · AIN · Metric · Distance

1 Introduction

In many scientific problems, numerical quantities are not known precisely but are specified with uncertainty. Such situations arise when values are obtained from imprecise measurements, aggregated expert assessments, or incomplete data [5]. A fundamental task is then the comparison of two uncertain quantities and the quantification of their dissimilarity. The natural mathematical formalization of this task is a metric, that is, a distance function satisfying non-negativity, identity of indiscernibles, symmetry, and the triangle inequality [3, 9].

A common representation of uncertain quantities is an interval defined by its lower and upper bounds. However, distances defined only in terms of interval bounds capture merely the support of the uncertainty [4]. In particular, interval-based metrics such as the Hausdorff distance assign distance zero to any two

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intervals with identical endpoints [1]. Consequently, two quantities that differ substantially in the location of their probability mass may be considered identical from the metric perspective. This limitation becomes evident when considering two uncertain quantities defined on the same interval but with different expected values. One quantity may concentrate probability near the lower bound, while the other concentrates near the upper bound. Despite representing substantially different uncertainty patterns, classical interval distances evaluate them as equal. Interval metrics therefore ignore the internal structure of uncertainty and cannot distinguish distributions sharing the same support.

Asymmetric Interval Numbers (AINs) address this limitation by augmenting an interval with an additional parameter representing the expected value [7]. Formally, an AIN is defined as a triple consisting of a lower bound, an upper bound, and an expected value. This representation can be interpreted as a parametric encoding of a probability distribution on the real line within a class of distributions with piecewise constant density with two levels. In this class, the density is uniquely determined by the normalization and expectation constraints.

Existing interval distance measures can formally be applied to AINs, yet they depend only on the interval endpoints. Consequently, they remain insensitive to the expected value and therefore to the distributional asymmetry encoded by AINs. Two AINs with identical support but different expectations are assigned distance zero by all classical interval metrics, despite representing different probability distributions.

To compare probability distributions rather than only their supports, a distribution-sensitive metric is required. Wasserstein metrics, originating from optimal transport theory [11], provide a natural framework for this purpose. For probability measures on the real line, the Wasserstein distance admits an equivalent formulation in terms of quantile functions, which makes it particularly suitable when quantile functions are known analytically. Moreover, unlike divergences such as the Kullback–Leibler divergence, Wasserstein distances remain well defined for distributions with non-overlapping supports [2].

The aim of this work is to construct mathematically rigorous and computationally efficient distance measures between asymmetric interval numbers based on Wasserstein metrics. We derive explicit analytical formulas for the distances W_1, W_2, and W_∞ and investigate their structural and metric properties in the space of AINs.

In particular, we address the following research questions:

- RQ1: Can explicit closed-form formulas for the Wasserstein distances W_1, W_2, and W_∞ between two AINs be derived from the piecewise-linear structure of their quantile functions?
- RQ2: Do the Wasserstein distances W_1, W_2, and W_∞, when restricted to the space of AINs, define proper metrics and preserve key structural properties required in practical use, such as consistency with real numbers, translation invariance, positive homogeneity, and the ordering relation $W_1 \leq W_2 \leq W_\infty$?

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RQ3: Can Wasserstein distances distinguish between AINs that share the same support interval but differ in the location of the expected value, in contrast to classical interval distances?

An additional consequence of the analytical formulas is computational simplicity. The distances W_1, W_2, and W_∞ can be evaluated directly from the three parameters of each AIN, without discretization of the underlying distributions or numerical integration of quantile functions. In contrast to numerical approximation schemes based on discretization, which incur computational cost $\mathcal{O}(n)$ in the number of sampling points, the proposed formulas compute all three distances in constant time $\mathcal{O}(1)$.

The remainder of the paper is organized as follows. Section 2 recalls the definition and probabilistic interpretation of asymmetric interval numbers. Section 3 introduces the Wasserstein distances and derives closed-form expressions for W_1, W_2, and W_∞ for AINs. Section 4 establishes the metric and structural properties of the proposed distances. Section 5 provides illustrative examples and comparisons. Finally, Section 6 concludes the paper.

2 Asymmetric Interval Numbers

This section recalls the definition and probabilistic interpretation of asymmetric interval numbers following [7].

Definition 1 (Asymmetric interval number). *An asymmetric interval number (AIN) is a triple $X = [a, b]_c$, where $a, b, c \in \mathbb{R}$ satisfy $a \le c \le b$, with $a < b$ for a non-degenerate AIN. The parameter c represents the expected value of the probability distribution induced by the AIN. When $a = b = c$, the AIN is called degenerate and corresponds to a real number.*

Every non-degenerate AIN $X = [a, b]_c$ induces a probability distribution on $[a, b]$ with a piecewise-constant density function:

$$f(x) = \begin{cases} \alpha, & a \le x < c, \\ \beta, & c \le x \le b, \\ 0, & \text{otherwise}, \end{cases} \tag{1}$$

where α and β are uniquely determined by the normalization condition $\int_a^b f(x)\,\mathrm{d}x = 1$ and the expectation constraint $\int_a^b x\,f(x)\,\mathrm{d}x = c$. Solving this system yields:

$$\alpha = \frac{b-c}{(b-a)(c-a)}, \qquad \beta = \frac{c-a}{(b-a)(b-c)}. \tag{2}$$

The cumulative distribution function of an AIN is given by:

$$F(x) = \begin{cases} 0, & x < a, \\ \alpha(x-a), & a \le x < c, \\ \alpha(c-a) + \beta(x-c), & c \le x \le b, \\ 1, & x > b. \end{cases} \tag{3}$$

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The quantile function (inverse CDF) of an AIN $[a, b]_c$ is piecewise linear with a single breakpoint at $t = \alpha(c - a)$:

$$Q(q) = \begin{cases} a + \dfrac{q}{\alpha}, & 0 \leq q \leq t, \\ c + \dfrac{q - t}{\beta}, & t \leq q \leq 1. \end{cases} \tag{4}$$

Remark 1. When $c = (a + b)/2$, the density satisfies $\alpha = \beta = 1/(b - a)$, and the AIN reduces to a classical interval number with a uniform distribution. In this sense, classical interval numbers form a symmetric subclass of AINs.

3 Wasserstein Distances for AINs

This section introduces the Wasserstein distances and derives closed-form expressions for W_1, W_2, and W_∞ between two asymmetric interval numbers.

3.1 Wasserstein distances

Let $\mathcal{P}(\mathbb{R})$ denote the set of all Borel probability measures on $\mathbb{R}$. For $p \geq 1$, the Wasserstein space of order p is defined as [10]:

$$\mathcal{P}_p(\mathbb{R}) = \left\{ P \in \mathcal{P}(\mathbb{R}) : \int_{\mathbb{R}} |x|^p \, \mathrm{d}P(x) < \infty \right\}. \tag{5}$$

For a probability measure $P \in \mathcal{P}_p(\mathbb{R})$, let F_P denote its cumulative distribution function and let Q_P denote its quantile function (generalized inverse):

$$Q_P(u) = \inf\{x \in \mathbb{R} : \; F_P(x) \geq u\}, \qquad u \in [0, 1]. \tag{6}$$

The Wasserstein distance of order p between two distributions in $\mathcal{P}_p(\mathbb{R})$ is given by [10, 11]:

$$W_p(X, Y) = \left(\int_0^1 |Q_X(u) - Q_Y(u)|^p \, \mathrm{d}u \right)^{\frac{1}{p}}. \tag{7}$$

The Wasserstein distance of order ∞, also called the Chebyshev–Wasserstein distance, is defined as:

$$W_\infty(X, Y) = \sup_{u \in [0,1]} |Q_X(u) - Q_Y(u)|. \tag{8}$$

Remark 2. Since every AIN induces a probability measure supported on the bounded interval $[a, b]$, the condition $\int_{\mathbb{R}} |x|^p \, \mathrm{d}P(x) < \infty$ is automatically satisfied for all $p \geq 1$. Therefore, AIN-induced distributions belong to $\mathcal{P}_p(\mathbb{R})$ for any $p \geq 1$, and the Wasserstein distances W_1, W_2, and W_∞ are well defined for all pairs of AINs.

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3.2 Setup and notation

Let $X = [a_1, b_1]_{c_1}$ and $Y = [a_2, b_2]_{c_2}$ be two AINs with distribution parameters α_i, β_i for $i = 1, 2$, as defined in (2). The quantile functions Q_X and Q_Y are piecewise linear with breakpoints at:

$$t_1 = \alpha_1(c_1 - a_1), \qquad t_2 = \alpha_2(c_2 - a_2). \tag{9}$$

Without loss of generality, assume $t_1 \leq t_2$. This can always be ensured by relabeling, since the Wasserstein distances are symmetric. The breakpoints partition the interval $[0, 1]$ into three subintervals:

$$[0, t_1], \quad [t_1, t_2], \quad [t_2, 1]. \tag{10}$$

On each subinterval, both quantile functions are linear, so their difference $D(q) = Q_X(q) - Q_Y(q)$ is also linear and can be written as $D_i(q) = A_i + B_i q$ for each subinterval, where the coefficients are:

Subinterval 1: $q \in [0, t_1]$, both quantiles in their left segment:

$$A_1 = a_1 - a_2, \qquad B_1 = \frac{1}{\alpha_1} - \frac{1}{\alpha_2}. \tag{11}$$

Subinterval 2: $q \in [t_1, t_2]$, Q_X in its right segment, Q_Y still in its left segment:

$$A_2 = c_1 - a_2 - \frac{t_1}{\beta_1}, \qquad B_2 = \frac{1}{\beta_1} - \frac{1}{\alpha_2}. \tag{12}$$

Subinterval 3: $q \in [t_2, 1]$, both quantiles in their right segment:

$$A_3 = c_1 - c_2 - \frac{t_1}{\beta_1} + \frac{t_2}{\beta_2}, \qquad B_3 = \frac{1}{\beta_1} - \frac{1}{\beta_2}. \tag{13}$$

3.3 Closed-form formula for W_1

Substituting $p = 1$ into (7):

$$W_1(X, Y) = \int_0^1 |Q_X(q) - Q_Y(q)| \, \mathrm{d}q = \sum_{i=1}^{3} \int_{p_i}^{r_i} |D_i(q)| \, \mathrm{d}q. \tag{14}$$

Since $D_i(q) = A_i + B_i q$ is linear on each subinterval $[p_i, r_i]$, it can change sign at most once. A sign change occurs if and only if:

$$\sigma_i = D_i(p_i) \cdot D_i(r_i) = (A_i + B_i p_i)(A_i + B_i r_i) < 0. \tag{15}$$

We denote the subinterval boundaries as:

$$p_1 = 0,\ r_1 = t_1, \qquad p_2 = t_1,\ r_2 = t_2, \qquad p_3 = t_2,\ r_3 = 1. \tag{16}$$

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Case 1: No sign change ($\sigma_i \geq 0$). The function D_i does not change sign on $[p_i, r_i]$, and:

$$I_i = \left| A_i(r_i - p_i) + \frac{B_i}{2}(r_i^2 - p_i^2) \right|. \tag{17}$$

Case 2: Sign change ($\sigma_i < 0$). The function D_i changes sign at:

$$q_{0,i} = -\frac{A_i}{B_i}, \tag{18}$$

and the integral splits into two parts:

$$I_i = \left| A_i(q_{0,i} - p_i) + \frac{B_i}{2}(q_{0,i}^2 - p_i^2) \right| + \left| A_i(r_i - q_{0,i}) + \frac{B_i}{2}(r_i^2 - q_{0,i}^2) \right|. \tag{19}$$

The W_1 distance is then:

$$W_1(X, Y) = I_1 + I_2 + I_3. \tag{20}$$

3.4 Closed-form formula for W_2

Substituting $p = 2$ into (7) and noting that $|D(q)|^2 = D(q)^2$:

$$W_2(X, Y) = \left(\int_0^1 (Q_X(q) - Q_Y(q))^2 \, \mathrm{d}q \right)^{\frac{1}{2}}. \tag{21}$$

The squaring operation eliminates the absolute value, so no sign-change analysis is required. The squared distance decomposes as:

$$W_2^2(X, Y) = \sum_{i=1}^{3} \int_{p_i}^{r_i} (A_i + B_i q)^2 \, \mathrm{d}q = \sum_{i=1}^{3} J_i, \tag{22}$$

where each term is evaluated by expanding $(A_i + B_i q)^2 = A_i^2 + 2A_iB_iq + B_i^2 q^2$ and integrating:

$$J_i = A_i^2(r_i - p_i) + A_iB_i(r_i^2 - p_i^2) + \frac{B_i^2}{3}(r_i^3 - p_i^3). \tag{23}$$

The W_2 distance is then:

$$W_2(X, Y) = \sqrt{J_1 + J_2 + J_3}. \tag{24}$$

Remark 3. In contrast to W_1, the formula for W_2 does not require sign-change analysis of the quantile difference $D(q)$, making it simpler both in formulation and in implementation.

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3.5 Closed-form formula for W_∞

The Chebyshev–Wasserstein distance is defined as:

$$W_\infty(X,Y) = \sup_{q\in[0,1]} |Q_X(q) - Q_Y(q)| = \sup_{q\in[0,1]} |D(q)|. \tag{25}$$

Since $D(q) = A_i + B_i q$ is linear on each subinterval $[p_i, r_i]$, and a linear function attains its extrema at the endpoints of its domain, the supremum of $|D(q)|$ over $[0,1]$ is attained at one of the subinterval boundaries. Therefore:

$$W_\infty(X,Y) = \max\{|D(0)|,\ |D(t_1)|,\ |D(t_2)|,\ |D(1)|\}, \tag{26}$$

where the values are computed directly from the coefficients:

$$D(0) = A_1 + B_1 \cdot 0 = a_1 - a_2, \tag{27}$$

$$D(t_1) = A_1 + B_1\, t_1 = A_2 + B_2\, t_1, \tag{28}$$

$$D(t_2) = A_2 + B_2\, t_2 = A_3 + B_3\, t_2, \tag{29}$$

$$D(1) = A_3 + B_3 \cdot 1 = b_1 - b_2. \tag{30}$$

Remark 4. The formula for W_∞ is the simplest among the three distances considered. It requires only four evaluations and no integration. The computational cost is $\mathcal{O}(1)$, as for W_1 and W_2.

Remark 5. The distance W_∞ measures the maximum pointwise difference between quantile functions, making it sensitive to local discrepancies between distributions. While W_1 and W_2 average the differences (without and with quadratic weighting, respectively), W_∞ is dominated by the worst-case quantile divergence. This property makes W_∞ particularly useful in applications where bounding the maximum comparison error is important.

4 Properties of W_1, W_2, and W_∞ for AINs

In this section, we investigate the fundamental properties of the Wasserstein distances W_1, W_2, and W_∞ defined on asymmetric interval numbers. We demonstrate that these distances inherit the metric axioms from the general theory of optimal transport, and we establish several additional properties relevant to their application in multi-criteria decision analysis.

Theorem 1 (Metric property). *The Wasserstein distances W_1, W_2, and W_∞ are proper metrics on the space of AINs, i.e., for any AINs X, Y, Z, the following axioms hold:*

1. *Non-negativity:* $W_p(X,Y) \ge 0$,
2. *Identity of indiscernibles:* $W_p(X,Y) = 0 \iff X = Y$,
3. *Symmetry:* $W_p(X,Y) = W_p(Y,X)$,
4. *Triangle inequality:* $W_p(X,Z) \le W_p(X,Y) + W_p(Y,Z)$,

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where $p \in \{1, 2, \infty\}$.

Proof. Every AIN $X = [a, b]_c$ induces a well-defined probability distribution μ_X on $\mathbb{R}$ with a piecewise-constant density function supported on the bounded interval $[a, b]$. The normalization condition $\int_a^b f(x)\,\mathrm{d}x = 1$ is satisfied by construction, and the boundedness of the support guarantees that $\mu_X \in \mathcal{P}_p(\mathbb{R})$ for all $p \geq 1$.

By the Kantorovich–Rubinstein theorem [11], the Wasserstein distance W_1 is a metric on $\mathcal{P}_1(\mathbb{R})$. More generally, for $p \geq 1$, the distance W_p is a metric on $\mathcal{P}_p(\mathbb{R})$ [11]. The distance W_∞ is a metric on the space of probability measures with bounded support [11]. Since the set of AIN-induced distributions forms a subset of these spaces, the restriction of W_p to this subset inherits all four metric axioms for $p \in \{1, 2, \infty\}$. □

Proposition 1 (Compatibility with real numbers). *If AINs degenerate to real numbers* $X = [x, x]_x$ *and* $Y = [y, y]_y$*, then*

$$W_1(X, Y) = W_2(X, Y) = W_\infty(X, Y) = |x - y|.$$

Proof. For a degenerate AIN $[x, x]_x$, the quantile function is constant: $Q_X(q) = x$ for all $q \in [0, 1]$. Therefore,

$$W_1(X, Y) = \int_0^1 |x - y|\,\mathrm{d}q = |x - y|,$$

$$W_2(X, Y) = \sqrt{\int_0^1 (x - y)^2\,\mathrm{d}q} = |x - y|,$$

$$W_\infty(X, Y) = \sup_{q \in [0,1]} |x - y| = |x - y|.$$

□

Remark 6. Proposition 1 confirms that the Wasserstein distances on AINs are consistent extensions of the standard Euclidean metric on $\mathbb{R}$.

Remark 7. More generally, the closed-form formulas derived in Section 3 remain valid when only one of the two AINs degenerates to a real number. Although the density-based representation from Section 2 requires $a < b$, the quantile formulation (7) extends naturally to all degenerate cases, where $Q_X(q) = x$ for all $q \in [0, 1]$.

Proposition 2 (Translation invariance). *For any AINs* X, Y *and any* $t \in \mathbb{R}$*,*

$$W_p(X + t, Y + t) = W_p(X, Y), \qquad p \in \{1, 2, \infty\}.$$

Proof. Translation shifts the quantile functions by a constant: $Q_{X+t}(q) = Q_X(q) + t$. The constants cancel in the difference:

$$Q_{X+t}(q) - Q_{Y+t}(q) = Q_X(q) - Q_Y(q).$$

Therefore, the integrals defining W_1 and W_2, as well as the supremum defining W_∞, remain unchanged. □

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Proposition 3 (Positive homogeneity). *For any AINs X, Y and any $k > 0$,*

$$W_p(kX, kY) = k\,W_p(X, Y), \qquad p \in \{1, 2, \infty\}.$$

Proof. Scaling an AIN by $k > 0$ scales the quantile function: $Q_{kX}(q) = k \cdot Q_X(q)$. For W_1:

$$W_1(kX, kY) = \int_0^1 |k \cdot Q_X(q) - k \cdot Q_Y(q)|\,\mathrm{d}q = k \int_0^1 |Q_X(q) - Q_Y(q)|\,\mathrm{d}q = k\,W_1(X, Y).$$

For W_2, the factor k exits the square root:

$$W_2(kX, kY) = \sqrt{\int_0^1 k^2 (Q_X(q) - Q_Y(q))^2\,\mathrm{d}q} = k\,W_2(X, Y).$$

For W_∞, the factor k exits the supremum:

$$\begin{aligned} W_\infty(kX, kY) &= \sup_{q \in [0,1]} |k \cdot Q_X(q) - k \cdot Q_Y(q)| \\ &= k \sup_{q \in [0,1]} |Q_X(q) - Q_Y(q)| = k\,W_\infty(X, Y). \end{aligned}$$

□

Proposition 4 (Sensitivity to uncertainty width). *If two AINs share the same expected value $c_1 = c_2$ but have different supports $(a_1, b_1) \neq (a_2, b_2)$, then*

$$W_p(X, Y) > 0, \qquad p \in \{1, 2, \infty\}.$$

Proof. Without loss of generality, assume $a_1 \neq a_2$. Since $Q_X(0) = a_1$ and $Q_Y(0) = a_2$, we have $D(0) = a_1 - a_2 \neq 0$. Therefore,

$$W_\infty(X, Y) \geq |D(0)| = |a_1 - a_2| > 0.$$

By continuity of the quantile functions, $D(q) \neq 0$ on a neighborhood of $q = 0$ of positive Lebesgue measure, so $W_1(X, Y) > 0$. The result for W_2 follows from the ordering $W_1 \leq W_2$ (Proposition 7). □

Proposition 5 (Sensitivity to asymmetry). *If two AINs have identical supports $(a_1, b_1) = (a_2, b_2)$ but different expected values $c_1 \neq c_2$, then*

$$W_p(X, Y) > 0, \qquad p \in \{1, 2, \infty\}.$$

Proof. Let $a_1 = a_2 = a$ and $b_1 = b_2 = b$. Since $c_1 \neq c_2$, the distribution parameters satisfy $\alpha_1 \neq \alpha_2$. On the subinterval $[0, \min\{t_1, t_2\}]$, both quantile functions are in their left segments, and

$$D(q) = q \left(\frac{1}{\alpha_1} - \frac{1}{\alpha_2} \right).$$

Since $\alpha_1 \neq \alpha_2$, we have $D(q) \neq 0$ for all $q > 0$ in this subinterval. The integral of $|D(q)|$ over a set of positive measure is strictly positive, hence $W_1(X, Y) > 0$. The results for W_2 and W_∞ follow from the ordering $W_1 \leq W_2 \leq W_\infty$ (Proposition 7). □

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Remark 8. Proposition 5 highlights a fundamental advantage of the Wasserstein distances over classical interval metrics. For two AINs with identical supports but different expected values, the Hausdorff distance yields $d_H(X, Y) = 0$, the width distance gives $|w(X) - w(Y)| = 0$, and the midpoint distance gives $|m(X) - m(Y)| = 0$. All of these classical measures operate solely on the interval endpoints, disregarding the internal distribution structure. The Wasserstein distances, by contrast, are sensitive to the shape of the distribution, and in particular to the asymmetry encoded by the expected value c.

Proposition 6 (Continuity). *If* $(a_n, b_n, c_n) \to (a, b, c)$*, then*

$$W_p([a_n, b_n]_{c_n},\, [a, b]_c) \to 0, \qquad p \in \{1, 2, \infty\}.$$

Proof. The quantile function $Q_{[a,b]_c}(q)$ is continuous in the parameters (a, b, c) for non-degenerate AINs, since the distribution parameters α and β are continuous functions of (a, b, c), and the quantile formula (4) is continuous in α, β, a, b, and c.

Therefore, $Q_{[a_n,b_n]_{c_n}}(q) \to Q_{[a,b]_c}(q)$ pointwise for all $q \in [0, 1]$. Moreover, for sufficiently large n, all quantile functions are uniformly bounded:

$$|Q_n(q) - Q(q)| \le M \quad \text{for all } q \in [0, 1],$$

where M is a constant depending on the convergent sequences. By the Lebesgue dominated convergence theorem,

$$W_1 = \int_0^1 |Q_n(q) - Q(q)|\, \mathrm{d}q \to 0,$$

and analogously $W_2^2 \to 0$. For W_∞, the convergence $Q_n \to Q$ is in fact uniform on the compact set $[0, 1]$, since the quantile functions are piecewise linear and their slopes and intercepts converge. Therefore,

$$W_\infty = \sup_{q \in [0,1]} |Q_n(q) - Q(q)| \to 0.$$

□

Remark 9. The continuity property has important practical implications. In multi-criteria decision analysis, where AIN parameters are elicited from expert judgments subject to natural imprecision, continuity guarantees that small perturbations in the input parameters (a, b, c) lead to proportionally small changes in the computed distances. This ensures the stability of resulting decision rankings. Furthermore, from a numerical standpoint, continuity guarantees robustness of the computations against floating-point rounding errors.

Theorem 2 (Symmetry of W_1 under symmetric shifts). *Let* $X = [a, b]_c$ *be an AIN with* $a < c < b$*, and let* $\delta \in (0, \min\{c-a, b-c\})$*. Define* $Y_- = [a, b]_{c-\delta}$ *and* $Y_+ = [a, b]_{c+\delta}$*. Then*

$$W_1(X, Y_-) = W_1(X, Y_+) = \delta.$$

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Proof. Since X, Y_-, and Y_+ share the same support $[a, b]$, we have $D(0) = 0$ and $D(1) = 0$ in all cases. The difference $D(q) = Q_X(q) - Q_Y(q)$ is a piecewise-linear function that starts and ends at zero. Its integral equals the difference of expected values:

$$\int_0^1 D(q)\,\mathrm{d}q = E(X) - E(Y).$$

We claim that when two AINs share the same support, $D(q)$ does not change sign on $[0, 1]$. Indeed, $D(0) = 0$ and, by the structure of AIN quantile functions, D is piecewise linear with at most three segments. Since $D(0) = 0$ and $D(1) = 0$, a sign change would require D to cross zero in the interior, which would imply the existence of a quantile q^* at which $Q_X(q^*) = Q_Y(q^*)$. However, for AINs with identical supports but $c_1 \neq c_2$, the quantile functions intersect only at the endpoints $q = 0$ and $q = 1$.

Therefore, $|D(q)| = D(q)$ or $|D(q)| = -D(q)$ throughout $[0, 1]$, and

$$W_1(X, Y) = \left| \int_0^1 D(q)\,\mathrm{d}q \right| = |c_1 - c_2|.$$

Applying this to Y_- and Y_+:

$$W_1(X, Y_-) = |c - (c - \delta)| = \delta = |c - (c + \delta)| = W_1(X, Y_+).$$

□

Theorem 3 (Symmetry condition for W_2). *Let $X = [a, b]_c$ with $c = \frac{a+b}{2}$ (i.e., X is a symmetric AIN), and let $\delta \in (0, \frac{b-a}{2})$. Define $Y_- = [a, b]_{c-\delta}$ and $Y_+ = [a, b]_{c+\delta}$. Then*

$$W_2(X, Y_-) = W_2(X, Y_+).$$

In general, for $c \neq \frac{a+b}{2}$, this equality does not hold.

Proof. When $c = \frac{a+b}{2}$, the AIN X has a symmetric distribution with $\alpha = \beta = \frac{2}{b-a}$. The quantile function Q_X is then symmetric about $q = \frac{1}{2}$ in the sense that $Q_X(q) + Q_X(1 - q) = a + b$ for all $q \in [0, 1]$.

Consider the substitution $u = 1-q$. Under this transformation, the symmetry of Q_X and the mirror relationship between Q_{Y_-} and Q_{Y_+} yield

$$(Q_X(q) - Q_{Y_-}(q))^2\big|_{q\to 1-u} = (Q_X(u) - Q_{Y_+}(u))^2,$$

and therefore

$$W_2^2(X, Y_-) = \int_0^1 (Q_X(q) - Q_{Y_-}(q))^2\,\mathrm{d}q = \int_0^1 (Q_X(u) - Q_{Y_+}(u))^2\,\mathrm{d}u = W_2^2(X, Y_+).$$

For $c \neq \frac{a+b}{2}$, the symmetry argument fails, and numerical counterexamples confirm that the equality does not hold in general (e.g., $X = [0, 10]_3$, $\delta = 1$: $W_2(X, Y_-) \neq W_2(X, Y_+)$). □

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Theorem 4 (Symmetry condition for W_∞). *Let $X = [a,b]_c$ with $c = \frac{a+b}{2}$, and let $\delta \in (0, \frac{b-a}{2})$. Define $Y_- = [a,b]_{c-\delta}$ and $Y_+ = [a,b]_{c+\delta}$. Then*

$$W_\infty(X, Y_-) = W_\infty(X, Y_+).$$

In general, for $c \neq \frac{a+b}{2}$, this equality does not hold.

Proof. When $c = \frac{a+b}{2}$, the quantile function Q_X is symmetric about $q = \frac{1}{2}$. By the same substitution $u = 1 - q$ as in the proof of Theorem 3:

$$|Q_X(q) - Q_{Y_-}(q)|\big|_{q \to 1-u} = |Q_X(u) - Q_{Y_+}(u)|.$$

Since the substitution $q \mapsto 1 - q$ is a bijection on $[0,1]$, the suprema coincide:

$$W_\infty(X, Y_-) = \sup_{q \in [0,1]} |Q_X(q) - Q_{Y_-}(q)| = \sup_{u \in [0,1]} |Q_X(u) - Q_{Y_+}(u)| = W_\infty(X, Y_+).$$

For $c \neq \frac{a+b}{2}$, the symmetry of Q_X does not hold, and the equality fails in general. □

Remark 10. The difference between W_1 on the one hand and W_2, W_∞ on the other in the context of symmetric shifts arises from the following observation. The distance W_1 depends only on the integral of $|D(q)|$, which for AINs with identical supports reduces to $|c_1 - c_2|$ regardless of the distribution shape. By contrast, W_2 involves the integral of $D(q)^2$ and W_∞ involves the supremum of $|D(q)|$, both of which are sensitive to the pointwise magnitude of the quantile difference and thus to the asymmetry of the reference distribution.

Proposition 7 (Magnitude bound). *For any AINs $X = [a_1, b_1]_{c_1}$ and $Y = [a_2, b_2]_{c_2}$,*

$$|c_1 - c_2| \leq W_1(X,Y) \leq W_2(X,Y) \leq W_\infty(X,Y) \leq d_H(X,Y),$$

where $d_H(X,Y) = \max\{|a_1 - a_2|, |b_1 - b_2|\}$ denotes the Hausdorff distance between the supports.

Proof. Lower bound. By Jensen's inequality applied to the convex function $|\cdot|$:

$$\begin{aligned} W_1(X,Y) = \int_0^1 |Q_X(q) - Q_Y(q)| \, \mathrm{d}q &\geq \left| \int_0^1 (Q_X(q) - Q_Y(q)) \, \mathrm{d}q \right| \\ &= |E(X) - E(Y)| = |c_1 - c_2|. \end{aligned}$$

Ordering $W_1 \leq W_2$. By Jensen's inequality applied to the convex function $\varphi(t) = t^2$:

$$\left(\int_0^1 |Q_X(q) - Q_Y(q)| \, \mathrm{d}q \right)^2 \leq \int_0^1 (Q_X(q) - Q_Y(q))^2 \, \mathrm{d}q,$$

which yields $W_1(X,Y)^2 \leq W_2(X,Y)^2$. Since both sides are non-negative, $W_1(X,Y) \leq W_2(X,Y)$.

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Ordering $W_2 \le W_\infty$. Since $|Q_X(q) - Q_Y(q)| \le W_\infty(X, Y)$ for all $q \in [0, 1]$:

$$W_2(X,Y) = \sqrt{\int_0^1 (Q_X(q) - Q_Y(q))^2 \, \mathrm{d}q} \le \sqrt{W_\infty(X,Y)^2 \int_0^1 \mathrm{d}q} = W_\infty(X,Y).$$

Upper bound $W_\infty \le d_H$. Since $Q_X(q) \in [a_1, b_1]$ and $Q_Y(q) \in [a_2, b_2]$ for all $q \in [0, 1]$:

$$|Q_X(q) - Q_Y(q)| \le \max\{|a_1 - a_2|, |b_1 - b_2|\} = d_H(X, Y).$$

Taking the supremum over q yields $W_\infty(X, Y) \le d_H(X, Y)$. □

Remark 11. The chain of inequalities $|c_1 - c_2| \le W_1 \le W_2 \le W_\infty \le d_H$ provides a complete hierarchy of distances between AINs. The Wasserstein distances occupy intermediate positions between the difference in expected values and the Hausdorff distance on the supports, combining information about the location of the distribution with information about its shape. The lower bound becomes an equality when the supports are identical (Theorem 2), while the upper bound is attained in the degenerate case.

5 Illustrative examples

We present three examples illustrating the behavior of the derived Wasserstein distances. All values are computed directly from the closed-form formulas of Section 3.

Example 1: Identical support, different asymmetry. Let $X = [0, 10]_3$ and $Y = [0, 10]_7$. Since the supports coincide, the Hausdorff distance equals zero: $d_H = 0$. Yet the two AINs represent qualitatively different uncertainty patterns, where X concentrates mass near the lower bound, Y near the upper. The Wasserstein distances capture this distinction: $W_1 = 4.00$, $W_2 \approx 4.43$, $W_\infty \approx 5.71$. The increasing values reflect a general property: W_1 measures the average quantile displacement, W_2 penalizes larger local deviations more heavily, and W_∞ reports the worst-case quantile divergence. In this example, the quantile functions diverge most near $q = 0$ and $q = 1$, which amplifies W_∞ relative to W_1.

Example 2: Full hierarchy on a mixed pair. Let $X = [1, 8]_3$ and $Y = [2, 6]_4$. Here the AINs differ in both support and asymmetry. The five quantities from Proposition 7 evaluate to:

$$|c_1 - c_2| = 1.00 \;\le\; W_1 \approx 1.30 \;\le\; W_2 \approx 1.36 \;\le\; W_\infty = 2.00 \;\le\; d_H = 2.00.$$

The gap between $|c_1 - c_2| = 1.00$ and $W_1 \approx 1.30$ shows that the distributional shape contributes to the distance beyond the shift in expected values alone. The equality $W_\infty = d_H = 2.00$ occurs because the maximum quantile divergence is attained at the endpoints ($|a_1 - a_2| = 1$, $|b_1 - b_2| = 2.00$), where the Wasserstein and Hausdorff distances coincide.

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Example 3: Symmetric vs. asymmetric reference. Let $\delta = 2$. We compare two reference AINs defined on $[0, 10]$ with different symmetry properties.

Symmetric reference. Let $X = [0, 10]_5$, $Y_1 = [0, 10]_3$, and $Y_2 = [0, 10]_7$. Both Y_1 and Y_2 are obtained by shifting the representative value by $\delta = 2$ in opposite directions. By Theorem 2, $W_1(X, Y_1) = W_1(X, Y_2) = 2.00$. Since X is symmetric ($c = (a + b)/2$), Theorem 3 further yields $W_2(X, Y_1) = W_2(X, Y_2) = 2.34$. The symmetric density treats both tails identically, so equal shifts produce equal distances under all three metrics.

Asymmetric reference. Let $X' = [0, 10]_3$, $Y_1' = [0, 10]_1$, and $Y_2' = [0, 10]_5$. Again, both are shifts of the representative value by $\delta = 2$. By Theorem 2, $W_1(X', Y_1') = W_1(X', Y_2') = 2.00$. However, since X' is asymmetric ($c \neq (a + b)/2$), the symmetry condition of Theorem 3 does not hold, and $W_2(X', Y_1') \neq W_2(X', Y_2')$. The shift toward the already-concentrated left tail ($c = 1$) produces a larger W_2 than the shift toward the sparser right tail ($c = 5$), because the quantile displacement is distributed less uniformly. This asymmetric response is invisible to W_1, which depends only on the total integral of the quantile difference, but is detected by W_2 through its sensitivity to the pointwise magnitude of the displacement.

6 Conclusion

This paper addressed the problem of measuring distances between asymmetric interval numbers using Wasserstein metrics. We now summarize the answers to the research questions posed in the introduction.

Regarding RQ1, we derived explicit closed-form analytical formulas for the Wasserstein distances W_1, W_2, and W_∞ between two AINs. The derivation exploits the piecewise-linear structure of the AIN quantile functions, which partitions the unit interval into at most three subintervals on each of which the quantile difference is linear. For W_1, the formula requires a sign-change analysis on each subinterval. For W_2, the squaring operation eliminates the absolute value, yielding a simpler expression. For W_∞, the distance reduces to the maximum of four point evaluations, requiring no integration at all. All three formulas are expressed directly in terms of the AIN parameters (a, b, c) and evaluate in constant time $\mathcal{O}(1)$.

Regarding RQ2, we proved that W_1, W_2, and W_∞ are proper metrics on the space of AINs, inheriting the metric axioms from the general theory of optimal transport. We further established compatibility with the Euclidean metric on real numbers, translation invariance, positive homogeneity, continuity with respect to AIN parameters, and the ordering relation $W_1 \leq W_2 \leq W_\infty$. The magnitude bound $|c_1 - c_2| \leq W_1 \leq W_2 \leq W_\infty \leq d_H$ places the Wasserstein distances within a complete hierarchy between the difference in expected values and the Hausdorff distance.

Regarding RQ3, we demonstrated that the Wasserstein distances are sensitive to distributional asymmetry. Two AINs with identical support but different expected values are assigned strictly positive distance by W_1, W_2, and

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W_∞, whereas the classical Hausdorff interval metric assigns distance zero. The Wasserstein distances are therefore more informative than the Hausdorff interval distance when comparing uncertain quantities represented as AINs.

The results presented in this work open several directions for future research. First, analogous closed-form formulas may be derivable for other distribution-sensitive distances, such as the Cramér distance and the Hellinger distance, exploiting the piecewise-constant density and piecewise-linear quantile structure of AINs. Second, the proposed Wasserstein distances can be integrated into multi-criteria decision analysis methods that require distance or dissimilarity measures between uncertain evaluations [8]. Third, the derived formulas can be implemented in the open-source Python library `asymintervals` [6], enabling their immediate use in scientific and engineering computations.

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