An Algorithm for Calculating the Multidimensional Solution of the Fuzzy Sylvester Matrix Equation

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Abstract. The paper presents a multidimensional horizontal approach to solving the fuzzy Sylvester matrix equation (FSME). The use of the horizontal membership function (HMF) of the fuzzy set allows for generating a granule of information about the FSME solution. The paper presents an algorithm for solving FSME using HMF, which generates a full FSME solution. The solution obtained using the given algorithm differs from the results presented in the cited articles. The calculated granule of the FSME solution contains solutions that do not occur in the results obtained in the analyzed examples, therefore these results are underestimated.

Keywords: Fuzzy Sylvester matrix equation · Horizontal membership function · Uncertainty theory · Artificial intelligence.

1 Introduction

Finding a solution to the fuzzy Sylvester matrix equation (FSME) is still being attempted by many scientists. The basic matrix Sylvester equation has the form $AX + XB = C$, where $A \in \mathbb{R}^{m\times m}$, $B \in \mathbb{R}^{m\times m}$, and $C \in \mathbb{R}^{n\times m}$, [5]. In this paper we will consider the FSME, which is the basic Sylvester matrix equation with fuzzy values. Fuzzy sets were proposed by Zadeh [11] and they model uncertainty in the form of a membership function $\mu : X \to [0, 1]$. The notation $\mu_A(x)$ represents the degree of membership of variable $x \in X$ to concept A.

Solutions to the FSME using different methods can be found in the articles [1–3, 9, 10]. In this paper, we propose the use of horizontal fuzzy set membership functions to solve the FSME. The horizontal membership function (HMF) of a fuzzy set was proposed by Piegat [8]. The HMF is a multidimensional approach to uncertainty modeling. It allows to represent the membership of a fuzzy set using a multidimensional information granule by introducing an additional variable called the relative distance measure (RDM) variable [7]. This theory is based on the notation of the interval as $\lambda x_1 + (1 - \lambda)x_2$, where $\lambda \in [0, 1]$, which is also described in the books [4, 12].

The main contribution of the article is the use of a multidimensional horizontal approach to the solution of the FSME, the proposal of an algorithm for

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calculating the solution of the FSME, and the demonstration that the methods presented in the cited articles do not provide a complete solution. The complete solution can be obtained by a method that does not rely only on the boundary values of the fuzzy number in arithmetic operations.

The rest of the paper is organized as follows: In Section 2, the theoretical background of FSME, horizontal membership function and the algorithm for calculating the FSME solution using HMF are presented. In Section 3, the FSME is calculated based on the proposed algorithm. In Section 4, conclusions from the presented research are given.

2 Theoretical Foundations

The fuzzy Sylvester matrix equation (FSME) has a form (1).

$$A\tilde{X} + \tilde{X}B = \tilde{C} \tag{1}$$

where $A = [a_{ij}] \in \mathbb{R}^{n\times n}$, $B = [b_{ij}] \in \mathbb{R}^{m\times m}$ and $\tilde{C} = [\tilde{c}_{ij}]$ is $n \times m$ matrix of fuzzy number, [2].

The equivalent form of the FSME has the form (2), [2].

$$(I_m \otimes A + B^T \otimes I_n)vec(\tilde{X}) = vec(\tilde{C}) \tag{2}$$

where I_k is $k \times k$ identity matrix, vec is a vectorization of matrix, $vec(\tilde{X})$ and $vec(\tilde{C})$ are $nm \times 1$ matrices, operation $\otimes$ is the Kronecker product, and B^T is transpose of a matrix B.

Definition 1. *[6, 8] (Horizontal membership function) Let $\tilde{u} : [a, b] \subseteq \mathbb{R} \to [0, 1]$ be a fuzzy number. The horizontal membership function $u^{gr} : [0, 1]\times[0, 1] \to [a, b]$ is a representation of $\tilde{u}(x)$ as $u^{gr}(\mu, \alpha_u) = x$ in which "gr" stands for the granule of information included in $x \in [a, b]$, $\mu \in [0, 1]$ is the membership degree of x in $\tilde{u}(x)$, $\alpha_u \in [0, 1]$ is called RDM variable, and $u^{gr}(\mu, \alpha_u) = \underline{u}^{\mu} + (\overline{u}^{\mu} - \underline{u}^{\mu})\alpha_u$.*

The horizontal membership function presented by granule of information of fuzzy number $\tilde{u}$ is denoted in [6] as $\mathcal{H}(\tilde{u}) = u^{gr}(\mu, \alpha_u)$.

Definition 2. *[6] The span of the information granule of the horizontal membership function $\mathcal{H}(\tilde{u}) = u^{gr}(\mu, \alpha_u)$ of $\tilde{u}(x) \in E_1$ is obtained as the μ-level sets of the vertical membership function of $\tilde{u}(x)$, using (12),*

$$\mathcal{H}^{-1}(u^{gr}(\mu, \alpha_u)) = \tilde{u}^{\mu} = \left[\inf_{\beta\geq\mu}\min_{\alpha_u} u^{gr}(\beta, \alpha_u), \sup_{\beta\geq\mu}\max_{\alpha_u} u^{gr}(\beta, \alpha_u)\right] \tag{3}$$

Let $\tilde{v}$ and $\tilde{w}$ are the fuzzy numbers, with the horizontal membership functions $\mathcal{H}(\tilde{v})$ and $\mathcal{H}(\tilde{w})$. Denotes $\odot$ as a one of four basic arithmetic operations $\{+, -, \times, /\}$. Therefore, $\tilde{v} \odot \tilde{w}$ is a fuzzy number such that $\mathcal{H}(\tilde{v} \odot \tilde{w}) = \mathcal{H}(\tilde{v}) \odot \mathcal{H}(\tilde{w})$, operation / occurs only if $0 \notin \mathcal{H}(\tilde{w})$, [6, 8].

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For arithmetic operations on fuzzy numbers with a horizontal membership function, the following basic algebraic properties hold, [6]:
1. $\tilde{u} - \tilde{v} = -(\tilde{v} - \tilde{u})$,
2. $\tilde{u} - \tilde{u} = 0$,
3. $\tilde{u}/\tilde{u} = 1$,
4. $(\tilde{u} + \tilde{v})\tilde{w} = \tilde{u}\tilde{w} + \tilde{v}\tilde{w}$.

Algorithm 1 presents the steps for calculating the solution of the fuzzy Sylvester matrix equation using horizontal membership function.

Algorithm 1 Algorithm for calculating the Sylvester fuzzy matrix equation $A_{n\times n}\tilde{X}_{n\times m} + \tilde{X}_{n\times m}B_{m\times m} = \tilde{C}_{n\times m}$

1. Check whether the matrices A and $-B$ do not share any eigenvalue. If they do, then STOP - fuzzy Sylvester matrix equation has no unique solution. Otherwise, Step 2.
2. Using the Kronecker product and the vectorization operator *vec* write the Sylvester equation in the form $(I_m \otimes A_{n\times n} + B^T_{m\times m} \otimes I_n)_{nm\times nm} vec(\tilde{X})_{nm\times 1} = vec(\tilde{C})_{nm\times 1}$, where I_k is $k \times k$ identity matrix.
3. Write down the elements of matrix $vec(\tilde{C})$ as arbitrary fuzzy numbers. Then rewrite the obtained equation using the horizontal membership function for matrix $vec(\tilde{C})$.
4. Calculate a granule of solution of the equation using basic arithmetic operations on fuzzy numbers with horizontal membership function.
5. Find the span of the granule of solution obtained in Step 4. Obtained fuzzy numbers represents minimum and maximum values of the multidimensional granule of solution.

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3 Numerical Examples

Example 1. (Example 4.3 in [2]) Let us consider a fuzzy Sylvester matrix equation $A\tilde{X} + \tilde{X}B = \tilde{C}^{\mu}$, as in (4).

$$\begin{aligned} &\begin{bmatrix} -1.2 & 0.2 \\ 0.8 & -1.2 \end{bmatrix} \begin{bmatrix} \tilde{x}_{11} & \tilde{x}_{12} \\ \tilde{x}_{21} & \tilde{x}_{22} \end{bmatrix} + \begin{bmatrix} \tilde{x}_{11} & \tilde{x}_{12} \\ \tilde{x}_{21} & \tilde{x}_{22} \end{bmatrix} \begin{bmatrix} 1.2 & 0.6 \\ 0.4 & 1.2 \end{bmatrix} \\ &= \begin{bmatrix} (120+12\mu, 144-12\mu) & (100+11\mu, 122-11\mu) \\ (100+12\mu, 124-12\mu) & (160+16\mu, 192-16\mu) \end{bmatrix} \end{aligned} \tag{4}$$

where $\mu \in [0, 1]$.

Considered the Sylvester equation (4) has a unique solution because matrices A and $-B$ do not share any eigenvalue.

Using the Kronecker product notation and the vectorization operator *vec* let us write the equation (4) in the form (5).

$$\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} -1.2 & 0.2 \\ 0.8 & -1.2 \end{bmatrix} + \begin{bmatrix} 1.2 & 0.6 \\ 0.4 & 1.2 \end{bmatrix}^T \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) vec(\tilde{X}) = vec(\tilde{C}^{\mu}) \tag{5}$$

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where

$$vec(\tilde{X}) = \begin{bmatrix} \tilde{x}_{11} \\ \tilde{x}_{21} \\ \tilde{x}_{12} \\ \tilde{x}_{22} \end{bmatrix}, vec(\tilde{C}^{\mu}) = \begin{bmatrix} (120+12\mu, 144-12\mu) \\ (100+12\mu, 124-12\mu) \\ (100+11\mu, 122-11\mu) \\ (160+16\mu, 192-16\mu) \end{bmatrix}, \mu \in [0,1].$$

Rewriting $vec(\tilde{C}^{\mu})$ using a horizontal membership function with variables μ, $\alpha_{c_{11}}$, $\alpha_{c_{12}}$, $\alpha_{c_{21}}$, $\alpha_{c_{22}} \in [0,1]$, we have multidimensional granule of information $vec(C^{gr})$ in the form:

$$vec(C^{gr}) = \begin{bmatrix} 120+12\mu+\alpha_{c_{11}}(24-24\mu) \\ 100+12\mu+\alpha_{c_{21}}(24-24\mu) \\ 100+11\mu+\alpha_{c_{12}}(22-22\mu) \\ 160+16\mu+\alpha_{c_{22}}(32-32\mu) \end{bmatrix}$$

With operation of Kronecker product and the multidimensional granular vector $vec(C^{gr})$ the equation (5) is equivalent to fuzzy matrix equation (6).

$$\begin{bmatrix} 0 & 0.2 & 0.4 & 0 \\ 0.8 & 0 & 0 & 0.4 \\ 0.6 & 0 & 0 & 0.2 \\ 0 & 0.6 & 0.8 & 0 \end{bmatrix} \begin{bmatrix} x_{11}^{gr} \\ x_{21}^{gr} \\ x_{12}^{gr} \\ x_{22}^{gr} \end{bmatrix} = \begin{bmatrix} 120+12\mu+\alpha_{c_{11}}(24-24\mu) \\ 100+12\mu+\alpha_{c_{21}}(24-24\mu) \\ 100+11\mu+\alpha_{c_{12}}(22-22\mu) \\ 160+16\mu+\alpha_{c_{22}}(32-32\mu) \end{bmatrix} \tag{6}$$

where $\mu, \alpha_{c_{11}}, \alpha_{c_{12}}, \alpha_{c_{21}}, \alpha_{c_{22}} \in [0,1]$

The solution to equation (6) is a multidimensional information granule of the form (7).

$$\begin{aligned} x_{11}^{gr}(\mu, \alpha_{c_{11}}, \alpha_{c_{12}}, \alpha_{c_{21}}, \alpha_{c_{22}}) &= 250+25\mu+(-60\alpha_{c_{21}}+110\alpha_{c_{12}})(1-\mu) \\ x_{21}^{gr}(\mu, \alpha_{c_{11}}, \alpha_{c_{12}}, \alpha_{c_{21}}, \alpha_{c_{22}}) &= -400-40\mu+(-240\alpha_{c_{11}}+160\alpha_{c_{22}})(1-\mu) \\ x_{12}^{gr}(\mu, \alpha_{c_{11}}, \alpha_{c_{12}}, \alpha_{c_{21}}, \alpha_{c_{22}}) &= 500+50\mu+(180\alpha_{c_{11}}-80\alpha_{c_{22}})(1-\mu) \\ x_{22}^{gr}(\mu, \alpha_{c_{11}}, \alpha_{c_{12}}, \alpha_{c_{21}}, \alpha_{c_{22}}) &= -250-20\mu+(180\alpha_{c_{21}}-220\alpha_{c_{12}})(1-\mu) \end{aligned} \tag{7}$$

It is easy to show that the obtained granular solution (7) satisfies the system of equations (4). Substituting the solution (7) into the matrix equation (4) we have:

$$\begin{aligned} L &= \begin{bmatrix} -1.2 & 0.2 \\ 0.8 & -1.2 \end{bmatrix} \begin{bmatrix} x_{11}^{gr}(\mu, \alpha_{c_{11}}, \alpha_{c_{12}}, \alpha_{c_{21}}, \alpha_{c_{22}}) & x_{12}^{gr}(\mu, \alpha_{c_{11}}, \alpha_{c_{12}}, \alpha_{c_{21}}, \alpha_{c_{22}}) \\ x_{21}^{gr}(\mu, \alpha_{c_{11}}, \alpha_{c_{12}}, \alpha_{c_{21}}, \alpha_{c_{22}}) & x_{22}^{gr}(\mu, \alpha_{c_{11}}, \alpha_{c_{12}}, \alpha_{c_{21}}, \alpha_{c_{22}}) \end{bmatrix} \\ &+ \begin{bmatrix} x_{11}^{gr}(\mu, \alpha_{c_{11}}, \alpha_{c_{12}}, \alpha_{c_{21}}, \alpha_{c_{22}}) & x_{12}^{gr}(\mu, \alpha_{c_{11}}, \alpha_{c_{12}}, \alpha_{c_{21}}, \alpha_{c_{22}}) \\ x_{21}^{gr}(\mu, \alpha_{c_{11}}, \alpha_{c_{12}}, \alpha_{c_{21}}, \alpha_{c_{22}}) & x_{22}^{gr}(\mu, \alpha_{c_{11}}, \alpha_{c_{12}}, \alpha_{c_{21}}, \alpha_{c_{22}}) \end{bmatrix} \begin{bmatrix} 1.2 & 0.6 \\ 0.4 & 1.2 \end{bmatrix} \\ &= \begin{bmatrix} 120+12\mu+\alpha_{c_{11}}(24-24\mu) & 100+11\mu+\alpha_{c_{12}}(22-22\mu) \\ 100+12\mu+\alpha_{c_{21}}(24-24\mu) & 160+16\mu+\alpha_{c_{22}}(32-32\mu) \end{bmatrix} \\ &= \begin{bmatrix} (120+12\mu, 144-12\mu) & (100+11\mu, 122-11\mu) \\ (100+12\mu, 124-12\mu) & (160+16\mu, 192-16\mu) \end{bmatrix} = R \end{aligned}$$

The span of the multidimensional granule of solution (7) in the form of matrix with fuzzy numbers is presented in equation (8), and also shown n Fig. 1, where

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$\alpha_{c_{ij}} = \{\alpha_{c_{11}}, \alpha_{c_{12}}, \alpha_{c_{21}}, \alpha_{c_{22}}\}$, and $\mu, \alpha_{c_{ij}} \in [0, 1]$.

$$\begin{aligned} &\begin{bmatrix} \mathcal{H}^{-1}(x_{11}^{gr}(\mu, \alpha_{c_{ij}})) & \mathcal{H}^{-1}(x_{12}^{gr}(\mu, \alpha_{c_{ij}})) \\ \mathcal{H}^{-1}(x_{21}^{gr}(\mu, \alpha_{c_{ij}})) & \mathcal{H}^{-1}(x_{22}^{gr}(\mu, \alpha_{c_{ij}})) \end{bmatrix} = \begin{bmatrix} \tilde{x}_{11}^{\mu} & \tilde{x}_{12}^{\mu} \\ \tilde{x}_{21}^{\mu} & \tilde{x}_{22}^{\mu} \end{bmatrix} \\ &= \begin{bmatrix} (190 + 85\mu, 360 - 85\mu) & (420 + 130\mu, 680 - 130\mu) \\ (-640 + 200\mu, -240 - 200\mu) & (-470 + 200\mu, -70 - 200\mu) \end{bmatrix} \end{aligned} \tag{8}$$

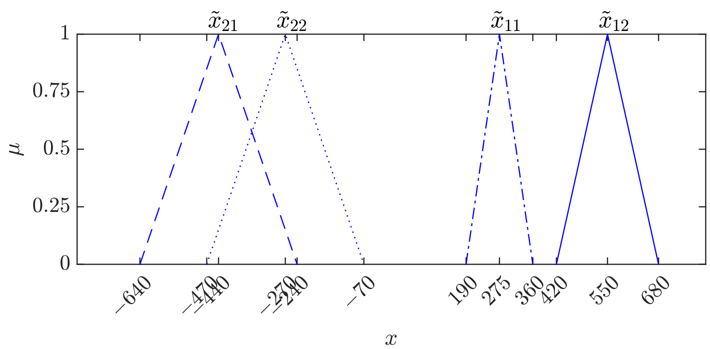

Fig. 1. Span of the multidimensional granule of solution (7).

The fuzzy numbers that are the result obtained in [2] are different from the fuzzy numbers (8) (also shown in Fig. 1). We will show that the solution in [2] is incomplete (underestimated). Let us consider one of the infinitely many solutions that can be generated from the multidimensional granule of solution (7), obtained using horizontal membership functions.

Let us consider the solution located on the support of solution (7). Substituting the values $\mu = \alpha_{c_{11}} = \alpha_{c_{12}} = \alpha_{c_{21}} = \alpha_{c_{22}} = 0$ with solution (7) we obtain the exact single solution (9).

$$\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} = \begin{bmatrix} 250 & 500 \\ -400 & -250 \end{bmatrix} \tag{9}$$

Single solution (9) satisfies Sylvester matrix equation (4). Substituting (9) into the left side of equation (4) gives us a matrix that is contained in the fuzzy matrix $\tilde{C}^{\mu}$ that is the right side of equation (4). This substitution operation is shown below.

$$\begin{aligned} L &= \begin{bmatrix} -1.2 & 0.2 \\ 0.8 & -1.2 \end{bmatrix} \begin{bmatrix} 250 & 500 \\ -400 & -250 \end{bmatrix} + \begin{bmatrix} 250 & 500 \\ -400 & -250 \end{bmatrix} \begin{bmatrix} 1.2 & 0.6 \\ 0.4 & 1.2 \end{bmatrix} \\ &= \begin{bmatrix} 120 & 100 \\ 100 & 160 \end{bmatrix} = \tilde{C}_{(\mu=\alpha_{c_{11}}=\alpha_{c_{12}}=\alpha_{c_{21}}=\alpha_{c_{22}}=0)} = R \end{aligned}$$

The exact single solution of (9) is not found in the solution given in Example 4.3 in [2]. This proves that the solution given in [2] is incomplete.

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Example 2. (Example 4.1 in [1]) Let us consider the fuzzy Sylvester matrix equation $A\tilde{X} + \tilde{X}B = \tilde{C}^{\mu}$, $\mu \in [0,1]$, (10).

$$\begin{aligned}&\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} \tilde{x}_{11} & \tilde{x}_{12} & \tilde{x}_{13} \\ \tilde{x}_{21} & \tilde{x}_{22} & \tilde{x}_{23} \end{bmatrix} + \begin{bmatrix} \tilde{x}_{11} & \tilde{x}_{12} & \tilde{x}_{13} \\ \tilde{x}_{21} & \tilde{x}_{22} & \tilde{x}_{23} \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & 1 \\ 1 & 2 & 0 \end{bmatrix} \\ &= \begin{bmatrix} (\mu, 2-\mu) & (-1+\mu, 1-\mu) & (1+\mu, 3-\mu) \\ (0, 1-\mu) & (0, 1-\mu) & (2\mu, 4-\mu) \end{bmatrix}\end{aligned} \tag{10}$$

Matrices A and $-B$ do not share any eigenvalue, so the Sylvester equation (10) has a unique solution. Using the Kronecker product's notation and the vectorization operator vec the equation (10) has a form of (11).

$$\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & 1 \\ 1 & 2 & 0 \end{bmatrix}^T \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) vec(\tilde{X}) = vec(\tilde{C}^{\mu}) \tag{11}$$

where

$$vec(\tilde{X}) = \begin{bmatrix} \tilde{x}_{11} \\ \tilde{x}_{21} \\ \tilde{x}_{12} \\ \tilde{x}_{22} \\ \tilde{x}_{13} \\ \tilde{x}_{23} \end{bmatrix}, vec(\tilde{C}^{\mu}) = \begin{bmatrix} (\mu, 2-\mu) \\ (0, 1-\mu) \\ (-1+\mu, 1-\mu) \\ (0, 1-\mu) \\ (1+\mu, 3-\mu) \\ (2\mu, 4-\mu) \end{bmatrix}, \mu \in [0,1].$$

With operation of Kronecker product and using a horizontal membership function with variables μ, $\alpha_{c_{11}}$, $\alpha_{c_{12}}$, $\alpha_{c_{13}}$, $\alpha_{c_{21}}$, $\alpha_{c_{22}}$, $\alpha_{c_{23}} \in [0,1]$ for the vector $vec(\tilde{C}^{\nu})$ the equation (11) is equivalent to fuzzy matrix equation (12).

$$\begin{bmatrix} 2 & 0 & 0 & 0 & 1 & 0 \\ -1 & 2 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 2 \\ -1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_{11}^{gr} \\ x_{21}^{gr} \\ x_{12}^{gr} \\ x_{22}^{gr} \\ x_{13}^{gr} \\ x_{23}^{gr} \end{bmatrix} = \begin{bmatrix} \mu + \alpha_{c_{11}}(2-2\mu) \\ \alpha_{c_{21}}(1-\mu) \\ -1+\mu+\alpha_{c_{12}}(2-2\mu) \\ \alpha_{c_{22}}(1-\mu) \\ 1+\mu+\alpha_{c_{13}}(2-2\mu) \\ 2\mu+\alpha_{c_{23}}(4-3\mu) \end{bmatrix} \tag{12}$$

The solution to equation (12) is a multidimensional information granule of the form (13).

$$\begin{aligned}
x_{11}^{gr}(\mu, \alpha_{c_{ij}}) &= 0.25 + 0.25\mu + (\alpha_{c_{11}} - 0.5\alpha_{c_{12}})(1-\mu) \\
x_{21}^{gr}(\mu, \alpha_{c_{ij}}) &= -0.3125 + 0.4375\mu + (0.25\alpha_{c_{11}} + 0.5\alpha_{c_{21}} + 0.125\alpha_{c_{12}} \\
&\quad -0.25\alpha_{c_{22}} - 0.5\alpha_{c_{13}})(1-\mu) \\
x_{12}^{gr}(\mu, \alpha_{c_{ij}}) &= 1.75 + 0.75\mu + (\alpha_{c_{11}} - 1.5\alpha_{c_{12}} + \alpha_{c_{13}})(1-\mu) \\
x_{22}^{gr}(\mu, \alpha_{c_{ij}}) &= -1.6875 + 2.0625\mu + (-0.25\alpha_{c_{11}} + 0.5\alpha_{c_{21}} + 1.875\alpha_{c_{12}} \\
&\quad -0.75\alpha_{c_{22}} - 1.5\alpha_{c_{13}})(1-\mu) + \alpha_{c_{23}}(4-3\mu) \\
x_{13}^{gr}(\mu, \alpha_{c_{ij}}) &= -0.5 + 0.5\mu + \alpha_{c_{12}}(1-\mu) \\
x_{23}^{gr}(\mu, \alpha_{c_{ij}}) &= 0.125 + 0.375\mu + (0.5\alpha_{c_{11}} - 0.75\alpha_{c_{12}} + 0.5\alpha_{c_{22}} + \alpha_{c_{13}}) \\
&\quad \cdot(1-\mu)
\end{aligned} \tag{13}$$

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where $\alpha_{c_{ij}} = \{\alpha_{c_{11}}, \alpha_{c_{12}}, \alpha_{c_{13}}, \alpha_{c_{21}}, \alpha_{c_{22}}, \alpha_{c_{23}}\}$, and $\mu, \alpha_{c_{ij}} \in [0,1]$.

It is easy to show that solution (13) satisfies fuzzy Sylvester matrix equation (10) and its equivalent forms. The proof is analogous to that of Example 1, so we leave it to the reader.

The span of the multidimensional solution granule (13) is presented in the form of fuzzy numbers, formula (14) and graphically in Fig. 2, where $\alpha_{c_{ij}}$, $\mu \in [0,1]$.

$$\begin{bmatrix} \mathcal{H}^{-1}(x_{11}^{gr}(\mu,\alpha_{c_{ij}})) & \mathcal{H}^{-1}(x_{12}^{gr}(\mu,\alpha_{c_{ij}})) & \mathcal{H}^{-1}(x_{13}^{gr}(\mu,\alpha_{c_{ij}})) \\ \mathcal{H}^{-1}(x_{21}^{gr}(\mu,\alpha_{c_{ij}})) & \mathcal{H}^{-1}(x_{22}^{gr}(\mu,\alpha_{c_{ij}})) & \mathcal{H}^{-1}(x_{23}^{gr}(\mu,\alpha_{c_{ij}})) \end{bmatrix} = \\ = \begin{bmatrix} \tilde{x}_{11}^{\mu} & \tilde{x}_{12}^{\mu} & \tilde{x}_{13}^{\mu} \\ \tilde{x}_{21}^{\mu} & \tilde{x}_{22}^{\mu} & \tilde{x}_{23}^{\mu} \end{bmatrix} = \begin{bmatrix} (-0.25+0.75\mu, 1.25-0.75\mu) & (0.25+2.25\mu, 4.75-2.25\mu) & (-0.5+0.5\mu, 0.5-0.5\mu) \\ (-1.0625+0.6875\mu, 0.5625-0.9375\mu) & (-4.1875+4.5625\mu, 4.6875-4.3125\mu) & (0.125+1.125\mu, 2.875-1.625\mu) \end{bmatrix} \tag{14}$$

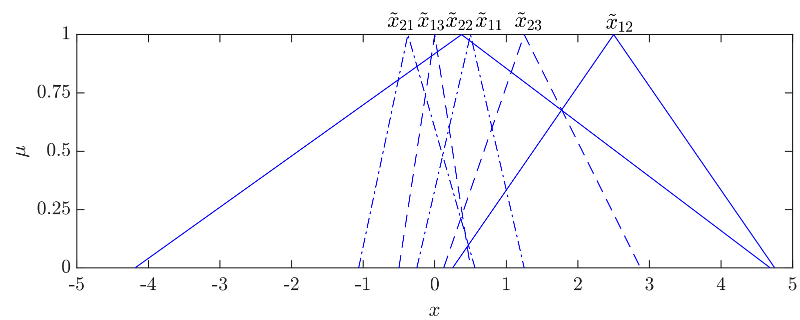

Fig. 2. Span of the multidimensional granule of solution (13).

To compare the obtained solution (13) with the solution from Example 4.1 in [1] let us generate an exact single solution. The exact single solution (15) is the solution obtained from the multidimensional solution (13), lying on support ($\mu = 0$), for variables $\alpha_{c_{11}} = \alpha_{c_{12}} = \alpha_{c_{13}} = \alpha_{c_{21}} = \alpha_{c_{22}} = \alpha_{c_{23}} = 0$.

$$\begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \end{bmatrix} = \begin{bmatrix} 0.25 & 1.75 & -0.5 \\ -0.3125 & -1.6875 & 0.875 \end{bmatrix} \tag{15}$$

The exact single solution (15) satisfies Sylvester matrix equation (10), the calculations are below.

$$L = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0.25 & 1.75 & -0.5 \\ -0.3125 & -1.6875 & 0.875 \end{bmatrix} + \begin{bmatrix} 0.25 & 1.75 & -0.5 \\ -0.3125 & -1.6875 & 0.875 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & 1 \\ 1 & 2 & 0 \end{bmatrix} \\ = \begin{bmatrix} 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \tilde{C}_{(\mu=\alpha_{c_{11}}=\alpha_{c_{12}}=\alpha_{c_{13}}=\alpha_{c_{21}}=\alpha_{c_{22}}=\alpha_{c_{23}}=0)} = R$$

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Single solution (15) is not included in the result obtained in Example 4.1 in [1]. This proves that the result obtained in [1] is not a full solution of the fuzzy Sylvester matrix equation (10).

4 Conclusions

The article presents an algorithm for calculating the solution of the FSME. The algorithm uses the horizontal membership function of the fuzzy set. The obtained solution of the FSME is a multidimensional information granule from which any single exact solution can be determined. The presented examples show that using the information granule, a solution can be generated that is not found in the results obtained by other methods from the cited articles. Application of the algorithm with HMF gives a full solution of the FSME.

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