# Multidimensional Granular Approach to Solving Fuzzy Complex System of Linear Equations

Marek Landowski

Faculty of Computer Science and Telecommunications, Maritime University of Szczecin, Waly Chrobrego 1-2, 70-500 Szczecin, Poland
m.landowski@pm.szczecin.pl

**Abstract.** The paper describes a multidimensional approach to solving a fuzzy complex linear system (FCLS). Together with the definition of arithmetic operations on complex fuzzy numbers with a horizontal granular membership function, the properties of these basic arithmetic operations are given and proven. Using the horizontal membership function of the fuzzy number and its counterpart in the complex number space, a multidimensional full granule of solution of FCLS was obtained. There are many methods in the scientific literature that generate results that are not full solutions of FCLS. The examples presented show that the use of the horizontal approach generates a full solution and indicate differences with the results obtained from other methods cited in the article. Furthermore, with granular approach the solution of the full FCLS was calculated.

**Keywords:** Fuzzy complex number · Horizontal complex fuzzy membership function · Fuzzy complex linear system · Uncertainty theory · Artificial intelligence.

## 1 Introduction

Fuzzy set was proposed by Zadeh [31]; subsequently, a fuzzy number as a fuzzy subset on the real line, with arithmetic on fuzzy numbers, called fuzzy arithmetic, were introduced [8, 10, 19, 22]. A fuzzy complex number (FCN) was defined by Backley [6, 7]; Ramot et al. [29, 30] also present concept of the FCN. In many papers the definition of FCN was used e.g. [3, 11, 13, 18, 28].

In this paper the multidimensional horizontal approach to the FCN is used to solve fuzzy complex system of linear equations. The horizontal membership function (HMF) of the fuzzy set and its arithmetic was introduced by Piegat for the first time was presented in [25]. The concept of the HMF is based on relative distance measure interval arithmetic (RDMIA) [24, 26, 27]. In this approach, the entire problem space is considered, where the solution is a multidimensional information granule. RDMIA uses the approach of defining the interval $[x_1, x_2]$ as a value of the form $\lambda x_1 + (1 - \lambda)x_2$, where $\lambda \in [0, 1]$. Such a definition of the interval was given by Zimmermann 1985 [32] and Klir and Yuan 1995 [20]. The

ICCS Camera Ready Version 2025
To cite this paper please use the final published version:
DOI: 10.1007/978-3-031-97573-8_14

HMF concept has been used in scientific works, e.g. for solving differential equations [21], as well as for studying the stability of fuzzy linear dynamic systems [23] and many others.

Determining full and correct solution of the fuzzy complex linear system (FCLS) is still very interesting for the research community. In problems of real life, variables usually are uncertain. For researchers of many fields of science, such as economy, medicine, engineering, mathematics or physics, it is very important to find a solution of the FCLS. The papers where the real fuzzy linear system is solved are [1, 2, 5, 9, 12, 16, 17]. Also, there are methods for solving FCLS and their application. Raheooy et al. in [28] applied FCLS to circuit analysis, while Jahantigh et al. in [18] proposed a numerical procedure to find the solution of FCLS. Behera and Chakraverty in [3, 4] describe a method for solving FCLS; later Ghanbari [13] published its modified version. Recently, Guo and Zhang in [14] wrote about minimal solution of FCLS. Farahani et al. in [11] solved FCLS using eigenvalue method. Moreover, Han and Guo in [15] discussed the numerical procedure for solution of FCLS.

The fuzzy complex linear system (FCLS) in [3] is defined as $[C][\tilde{Z}] = [\tilde{W}]$, where: $[C]$ - the $n \times n$ matrix of crisp complex coefficients; $[\tilde{W}]$ - the $n \times 1$ matrix of fuzzy complex numbers; and $[\tilde{Z}]$ - unknown fuzzy complex numbers. The full fuzzy complex linear system (FFCLS) is defined as a FCLS where the matrix of coefficients $[C]$ is a matrix of fuzzy complex numbers $[\tilde{C}]$.

On the presented examples it was shown that results generated by methods in the cited papers are not full solutions of the FCLS. It was proved by generating the crisp solution of the crisp complex linear system (CCLS) from FCLS which does not belong to the set of results of the cited papers. Therefore, there is a need to find a method that will generate a full solution of the FCLS. To fill this gap, the HMF and its arithmetic is used.

The method with fuzzy complex number (FCN) and its horizontal membership function (HMF) presented in the paper allows to generate crisp sets of the dependent solutions, the granule of solution. With the solution expressed in the form of granule of information it can be determined if any crisp solution is possible or not. Three examples analyzed in the paper were taken from [3, 4, 15, 11, 28]. The last example presents application of the HMF to find the solution of the full fuzzy complex linear system.

The multidimensional granular complex HMF approach provides solutions that satisfy fuzzy complex linear systems. It is shown on examples that the methods from the cited articles do not generate a full FCLS solution, the results are overestimated or underestimated. The obtained multidimensional solution by HMF is an information granule of the full set of solutions that satisfies FCLS (FFCLS) and its equivalent forms.

The rest of the paper is organized as follows: in Section 2 the theoretical foundation of the fuzzy complex number with horizontal membership function and its arithmetic with basic algebraic properties are given. In Section 3 the numerical examples are considered, where the fuzzy complex linear systems are solved. Section 4 presents some conclusions.

ICCS Camera Ready Version 2025
To cite this paper please use the final published version:
DOI: 10.1007/978-3-031-97573-8_14

Abbreviations: FCN - fuzzy complex number, CCLS - crisp complex linear system, FCLS - fuzzy complex linear system, FFCLS - full fuzzy complex linear system, HMF - horizontal membership function.

## 2 Theoretical Foundations

This section presents the theoretical foundations of the fuzzy complex number, horizontal fuzzy number, horizontal fuzzy complex number and horizontal fuzzy complex arithmetic used in the paper for solving fuzzy complex linear system.

Let us introduce the following notations: $E_1$ is a set of fuzzy numbers on the real numbers $\mathbb{R}$, and $u^{\mu}_{\tilde{x}} = (\underline{u}^{\mu}_{\tilde{x}}, \overline{u}^{\mu}_{\tilde{x}})$ is $\mu$-cut level of $\tilde{x} \in E_1$, $\mu \in [0,1]$.

**Definition 1.** *[12] (Fuzzy number) An arbitrary fuzzy number is an ordered pair of functions* $(\underline{u}^{\mu}, \overline{u}^{\mu})$, $0 \leq \mu \leq 1$, *which satisfy the following requirements:*

1. $\underline{u}^{\mu}$ *is a bounded left continuous nondecreasing function over* $[0,1]$.
2. $\overline{u}^{\mu}$ *is a bounded left continuous nonincreasing function over* $[0,1]$.
3. $\underline{u}^{\mu} \leq \overline{u}^{\mu}$, $0 \leq \mu \leq 1$.

**Definition 2.** *[3] (Fuzzy complex number) An arbitrary fuzzy complex number is written as* $\tilde{z} = \tilde{p} + i\tilde{q}$, *where* $\tilde{p} = [\underline{p}^{\mu}, \overline{p}^{\mu}]$ *and* $\tilde{q} = [\underline{q}^{\mu}, \overline{q}^{\mu}]$, $0 \leq \mu \leq 1$.

For fuzzy complex number $\tilde{z} = \tilde{p} + i\tilde{q}$, $\tilde{p}$ is called the real part $(Re(\tilde{z}))$ and $\tilde{q}$ is called the imaginary part $(Im(\tilde{z}))$. A $\mu$-level of an arbitrary fuzzy complex number can be written as $\tilde{z}^{\mu} = [\underline{p}^{\mu}, \overline{p}^{\mu}] + i[\underline{q}^{\mu}, \overline{q}^{\mu}] = [\underline{p}^{\mu} + i\underline{q}^{\mu}, \overline{p}^{\mu} + i\overline{q}^{\mu}]$.

**Definition 3.** *[21, 25] (Horizontal membership function) Let* $\tilde{u} : [a,b] \subseteq \mathbb{R} \rightarrow [0,1]$ *be a fuzzy number. The horizontal membership function* $u^{gr} : [0,1] \times [0,1] \rightarrow [a,b]$ *is a representation of* $\tilde{u}(x)$ *as* $u^{gr}(\mu, \alpha_u) = x$ *in which "gr" stands for the granule of information included in* $x \in [a,b]$, $\mu \in [0,1]$ *is the membership degree of* $x$ *in* $\tilde{u}(x)$, $\alpha_u \in [0,1]$ *is called RDM variable, and* $u^{gr}(\mu, \alpha_u) = \underline{u}^{\mu} + (\overline{u}^{\mu} - \underline{u}^{\mu})\alpha_u$.

The horizontal membership function presented by granule of information of fuzzy number $\tilde{u}$ is denoted in [21] as $\mathcal{H}(\tilde{u}) = u^{gr}(\mu, \alpha_u)$.

**Definition 4.** *[23] The horizontal membership function of the fuzzy complex number* $\tilde{z} = \tilde{p} + i\tilde{q}$ *is denoted by* $\mathcal{H}(\tilde{z}) = z^{gr}(\mu, \alpha_p, \alpha_q)$ *and defined as* $\mathcal{H}(\tilde{z}) = \mathcal{H}(\tilde{p}) + i\mathcal{H}(\tilde{q})$. *Therefore,* $Re(\mathcal{H}(\tilde{z})) = \mathcal{H}(Re(\tilde{z}))$ *and* $Im(\mathcal{H}(\tilde{z})) = \mathcal{H}(Im(\tilde{z}))$.

For example, the fuzzy complex number with nonlinear borders $\tilde{z}^{\mu} = \tilde{p}^{\mu} + i\tilde{q}^{\mu} = [3 + \mu^2, 5 - \sqrt{\mu}] + i[4 + \mu^3, 7 - \mu^2]$, $\mu \in [0,1]$, has a complex granular form of membership function as: $z^{gr}(\mu, \alpha_p, \alpha_q) = 3 + \mu^2 + \alpha_p(2 - \sqrt{\mu} - \mu^2) + i[4 + \mu^3 + \alpha_q(3 - \mu^2 - \mu^3)]$, where $\mu$, $\alpha_p$, $\alpha_q \in [0,1]$. The fuzzy complex number $\tilde{z}^{\mu}$ with nonlinear boundaries together with variables $\alpha_p$ and $\alpha_q$ generating the information granule is presented in 3D space in Fig. 1.

The definition of the span of the complex information granule $z^{gr}(\mu, \alpha_p, \alpha_q)$ is as follows.

ICCS Camera Ready Version 2025
To cite this paper please use the final published version:
DOI: 10.1007/978-3-031-97573-8_14

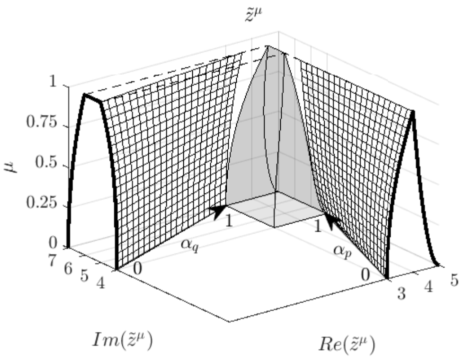


**Fig. 1.** Fuzzy complex number $\tilde{z}^\mu = \tilde{p}^\mu + i\tilde{q}^\mu = [3+\mu^2, 5-\sqrt{\mu}] + i[4+\mu^3, 7-\mu^2]$, $\mu \in [0,1]$ and variables $\alpha_p, \alpha_q \in [0,1]$ generating the information granule.

**Definition 5.** *The fuzzy complex number $\tilde{z} = \tilde{p} + i\tilde{q}$ can be represented by the span of its horizontal membership function $\mathcal{H}(\tilde{z}) = z^{gr}(\mu, \alpha_p, \alpha_q)$, by the formula (1),*

$$\begin{aligned}\tilde{z}^\mu = \mathcal{H}^{-1}(z^{gr}(\mu,\alpha_p,\alpha_q)) = &\left[\inf_{\gamma\geq\mu}\min_{\alpha_p} z^{gr}(\gamma,\alpha_p), \sup_{\gamma\geq\mu}\max_{\alpha_p} z^{gr}(\gamma,\alpha_p)\right] \\ &+i\left[\inf_{\gamma\geq\mu}\min_{\alpha_q} z^{gr}(\gamma,\alpha_q), \sup_{\gamma\geq\mu}\max_{\alpha_q} z^{gr}(\gamma,\alpha_q)\right]\end{aligned} \tag{1}$$

**Definition 6.** *Let $\tilde{z}_1 = \tilde{p}_1 + i\tilde{q}_1$ and $\tilde{z}_2 = \tilde{p}_2 + i\tilde{q}_2$ are the fuzzy complex numbers, with the horizontal membership functions $\mathcal{H}(\tilde{z}_1) = \mathcal{H}(\tilde{p}_1) + i\mathcal{H}(\tilde{q}_1)$ and $\mathcal{H}(\tilde{z}_2) = \mathcal{H}(\tilde{p}_2) + i\mathcal{H}(\tilde{q}_2)$. Denotes $\odot$ as a one of four basic arithmetic operations $\{+,-,\times,/\}$. Therefore, $\tilde{z}_1 \odot \tilde{z}_2$ is a fuzzy complex number such that $\mathcal{H}(\tilde{z}_1 \odot \tilde{z}_2) = \mathcal{H}(\tilde{p}_1) \odot \mathcal{H}(\tilde{p}_2) + i(\mathcal{H}(\tilde{q}_1) \odot \mathcal{H}(\tilde{q}_2))$, operation / occurs only if $0 \notin \mathcal{H}(\tilde{z}_2)$*

For fuzzy numbers whose horizontal membership functions are as in Definition 4 the basic algebraic properties hold.

**Lemma 1.** *Let $\tilde{z}_1$, $\tilde{z}_2$ and $\tilde{z}_3$ are fuzzy complex numbers with horizontal membership functions $\mathcal{H}(\tilde{z}_1)$, $\mathcal{H}(\tilde{z}_2)$ and $\mathcal{H}(\tilde{z}_3)$. The FCN with HMF addition and multiplication are commutative and associative, equations in (2) are true,*

$$\begin{aligned}&\tilde{z}_1 + \tilde{z}_2 = \tilde{z}_2 + \tilde{z}_1, \\ &\tilde{z}_1\tilde{z}_2 = \tilde{z}_2\tilde{z}_1, \\ &\tilde{z}_1 + (\tilde{z}_2 + \tilde{z}_3) = (\tilde{z}_1 + \tilde{z}_2) + \tilde{z}_3, \\ &\tilde{z}_1(\tilde{z}_2\tilde{z}_3) = (\tilde{z}_1\tilde{z}_2)\tilde{z}_3.\end{aligned} \tag{2}$$

**Lemma 2.** *The FCN with HMF has identity elements under addition and multiplication that are crisp values 0 and 1, respectively. Equations in (3) hold for any FCN $\tilde{z}$ with HMF $\mathcal{H}(\tilde{z})$,*

$$\begin{aligned}&0 + \tilde{z} = \tilde{z} + 0 = \tilde{z}, \\ &1 \cdot \tilde{z} = \tilde{z} \cdot 1 = \tilde{z}.\end{aligned} \tag{3}$$

ICCS Camera Ready Version 2025
To cite this paper please use the final published version:
DOI: 10.1007/978-3-031-97573-8_14

It is easy to show that Lemma 1 and 2 hold, so the proof is left to the reader.

**Lemma 3.** *The FCN* $\tilde{z} = \tilde{p} + i\tilde{q}$ *with HMF* $\mathcal{H}(\tilde{z}) = z^{gr}(\mu, \alpha_p, \alpha_q) = \underline{p}^{\mu} + \alpha_p(\overline{p}^{\mu} - \underline{p}^{\mu}) + i[\underline{q}^{\mu} + \alpha_q(\overline{q}^{\mu} - \underline{q}^{\mu}]$, *where* $\mu, \alpha_p, \alpha_q \in [0, 1]$, *has an additive inverse element (4),*

$$-z^{gr}(\mu, \alpha_p, \alpha_q) = -[\underline{p}^{\mu} + \alpha_p(\overline{p}^{\mu} - \underline{p}^{\mu})] - i[\underline{q}^{\mu} + \alpha_q(\overline{q}^{\mu} - \underline{q}^{\mu})]. \tag{4}$$

*Proof.* $z^{gr}(\mu, \alpha_p, \alpha_q) + (-z^{gr}(\mu, \alpha_p, \alpha_q)) = [\underline{p}^{\mu} + \alpha_p(\overline{p}^{\mu} - \underline{p}^{\mu}) + i[\underline{q}^{\mu} + \alpha_q(\overline{q}^{\mu} - \underline{q}^{\mu})]] + [-[\underline{p}^{\mu} + \alpha_p(\overline{p}^{\mu} - \underline{p}^{\mu})] - i[\underline{q}^{\mu} + \alpha_q(\overline{q}^{\mu} - \underline{q}^{\mu})]] = \underline{p}^{\mu} + \alpha_p(\overline{p}^{\mu} - \underline{p}^{\mu}) + i[\underline{q}^{\mu} + \alpha_q(\overline{q}^{\mu} - \underline{q}^{\mu})] - [\underline{p}^{\mu} + \alpha_p(\overline{p}^{\mu} - \underline{p}^{\mu})] - i[\underline{q}^{\mu} + \alpha_q(\overline{q}^{\mu} - \underline{q}^{\mu})] = 0$ □

**Lemma 4.** *For the FCN* $\tilde{z} = \tilde{p} + i\tilde{q}$ *with HMF* $H(\tilde{z}) = z^{gr}(\mu, \alpha_p, \alpha_q) = \underline{p}^{\mu} + \alpha_p(\overline{p}^{\mu} - \underline{p}^{\mu}) + i[\underline{q}^{\mu} + \alpha_q(\overline{q}^{\mu} - \underline{q}^{\mu})]$, $0 \notin z^{gr}(\mu, \alpha_p, \alpha_q)$, *where* $\mu, \alpha_p, \alpha_q \in [0, 1]$, *there exists a multiplicative inverse element given by (5),*

$$1/z^{gr}(\mu, \alpha_p, \alpha_q) = \frac{\underline{p}^{\mu} + \alpha_p(\overline{p}^{\mu} - \underline{p}^{\mu}) - i[\underline{q}^{\mu} + \alpha_q(\overline{q}^{\mu} - \underline{q}^{\mu})]}{[\underline{p}^{\mu} + \alpha_p(\overline{p}^{\mu} - \underline{p}^{\mu})]^2 + [\underline{q}^{\mu} + \alpha_q(\overline{q}^{\mu} - \underline{q}^{\mu})]^2}. \tag{5}$$

*Proof.* $z^{gr}(\mu, \alpha_p, \alpha_q) \cdot (1/z^{gr}(\mu, \alpha_p, \alpha_q))$
$= [\underline{p}^{\mu} + \alpha_p(\overline{p}^{\mu} - \underline{p}^{\mu}) + i[\underline{q}^{\mu} + \alpha_q(\overline{q}^{\mu} - \underline{q}^{\mu})]] \cdot \frac{\underline{p}^{\mu} + \alpha_p(\overline{p}^{\mu} - \underline{p}^{\mu}) - i[\underline{q}^{\mu} + \alpha_q(\overline{q}^{\mu} - \underline{q}^{\mu})]}{[\underline{p}^{\mu} + \alpha_p(\overline{p}^{\mu} - \underline{p}^{\mu})]^2 + [\underline{q}^{\mu} + \alpha_q(\overline{q}^{\mu} - \underline{q}^{\mu})]^2}$
$= \frac{[\underline{p}^{\mu} + \alpha_p(\overline{p}^{\mu} - \underline{p}^{\mu})]^2 + [\underline{q}^{\mu} + \alpha_q(\overline{q}^{\mu} - \underline{q}^{\mu})]^2}{[\underline{p}^{\mu} + \alpha_p(\overline{p}^{\mu} - \underline{p}^{\mu})]^2 + [\underline{q}^{\mu} + \alpha_q(\overline{q}^{\mu} - \underline{q}^{\mu})]^2} = 1$ □

**Lemma 5.** *The distributive law (6) holds for any FCN* $\tilde{z}_1 = \tilde{p}_1 + i\tilde{q}_1$, $\tilde{z}_2 = \tilde{p}_2 + i\tilde{q}_2$ *and* $\tilde{z}_3 = \tilde{p}_3 + i\tilde{q}_3$ *with* $\mathcal{H}(\tilde{z}_1) = z_1^{gr}(\mu, \alpha_{p_1}, \alpha_{q_1})$, $\mathcal{H}(\tilde{z}_2) = z_2^{gr}(\mu, \alpha_{p_2}, \alpha_{q_2})$ *and* $\mathcal{H}(\tilde{z}_3) = z_3^{gr}(\mu, \alpha_{p_3}, \alpha_{q_3})$,

$$\tilde{z}_1(\tilde{z}_2 + \tilde{z}_3) = \tilde{z}_1\tilde{z}_2 + \tilde{z}_1\tilde{z}_3 \tag{6}$$

*Proof.* $\mathcal{H}(\tilde{z}_1)(\mathcal{H}(\tilde{z}_2) + \mathcal{H}(\tilde{z}_3)) = [\underline{p}_1^{\mu} + \alpha_{p_1}(\overline{p}_1^{\mu} - \underline{p}_1^{\mu}) + i[\underline{q}_1^{\mu} + \alpha_{q_1}(\overline{q}_1^{\mu} - \underline{q}_1^{\mu})]] \cdot \{[\underline{p}_2^{\mu} + \alpha_{p_2}(\overline{p}_2^{\mu} - \underline{p}_2^{\mu}) + i[\underline{q}_2^{\mu} + \alpha_{q_2}(\overline{q}_2^{\mu} - \underline{q}_2^{\mu})]] + [\underline{p}_3^{\mu} + \alpha_{p_3}(\overline{p}_3^{\mu} - \underline{p}_3^{\mu}) + i[\underline{q}_3^{\mu} + \alpha_{q_3}(\overline{q}_3^{\mu} - \underline{q}_3^{\mu})]]\} = [\underline{p}_1^{\mu} + \alpha_{p_1}(\overline{p}_1^{\mu} - \underline{p}_1^{\mu}) + i[\underline{q}_1^{\mu} + \alpha_{q_1}(\overline{q}_1^{\mu} - \underline{q}_1^{\mu})]] \cdot [\underline{p}_2^{\mu} + \alpha_{p_2}(\overline{p}_2^{\mu} - \underline{p}_2^{\mu}) + i[\underline{q}_2^{\mu} + \alpha_{q_2}(\overline{q}_2^{\mu} - \underline{q}_2^{\mu})]] + [\underline{p}_1^{\mu} + \alpha_{p_1}(\overline{p}_1^{\mu} - \underline{p}_1^{\mu}) + i[\underline{q}_1^{\mu} + \alpha_{q_1}(\overline{q}_1^{\mu} - \underline{q}_1^{\mu})]] \cdot [\underline{p}_3^{\mu} + \alpha_{p_3}(\overline{p}_3^{\mu} - \underline{p}_3^{\mu}) + i[\underline{q}_3^{\mu} + \alpha_{q_3}(\overline{q}_3^{\mu} - \underline{q}_3^{\mu})]] = \mathcal{H}(\tilde{z}_1)\mathcal{H}(\tilde{z}_2) + \mathcal{H}(\tilde{z}_1)\mathcal{H}(\tilde{z}_3)$,
where $\mu, \alpha_{p_1}, \alpha_{p_2}, \alpha_{p_3}, \alpha_{q_1}, \alpha_{q_2}, \alpha_{q_3} \in [0, 1]$. □

**Lemma 6.** *The cancellation law for addition (7) holds in the FCN with HMF arithmetic,*

$$\tilde{z}_1 + \tilde{z}_3 = \tilde{z}_2 + \tilde{z}_3 \Rightarrow \tilde{z}_1 = \tilde{z}_2, \tag{7}$$

*where* $\tilde{z}_1 = \tilde{p}_1 + i\tilde{q}_1$, $\tilde{z}_2 = \tilde{p}_2 + i\tilde{q}_2$ *and* $\tilde{z}_3 = \tilde{p}_3 + i\tilde{q}_3$ *are FCN with HMF:* $\mathcal{H}(\tilde{z}_1) = z_1^{gr}(\mu, \alpha_{p_1}, \alpha_{q_1})$, $\mathcal{H}(\tilde{z}_2) = z_2^{gr}(\mu, \alpha_{p_2}, \alpha_{q_2})$ *and* $\mathcal{H}(\tilde{z}_3) = z_3^{gr}(\mu, \alpha_{p_3}, \alpha_{q_3})$.

*Proof.* $\tilde{z}_1 + \tilde{z}_3 = \tilde{z}_2 + \tilde{z}_3$;
$[\underline{p}_1^{\mu} + \alpha_{p_1}(\overline{p}_1^{\mu} - \underline{p}_1^{\mu}) + i[\underline{q}_1^{\mu} + \alpha_{q_1}(\overline{q}_1^{\mu} - \underline{q}_1^{\mu})]] + [\underline{p}_3^{\mu} + \alpha_{p_3}(\overline{p}_3^{\mu} - \underline{p}_3^{\mu}) + i[\underline{q}_3^{\mu} + \alpha_{q_3}(\overline{q}_3^{\mu} - \underline{q}_3^{\mu})]] = [\underline{p}_2^{\mu} + \alpha_{p_2}(\overline{p}_2^{\mu} - \underline{p}_2^{\mu}) + i[\underline{q}_2^{\mu} + \alpha_{q_2}(\overline{q}_2^{\mu} - \underline{q}_2^{\mu})]] + [\underline{p}_3^{\mu} + \alpha_{p_3}(\overline{p}_3^{\mu} - \underline{p}_3^{\mu}) + i[\underline{q}_3^{\mu} + \alpha_{q_3}(\overline{q}_3^{\mu} - \underline{q}_3^{\mu})]]$;
Adding $-\tilde{z}_3$ on both sides, we have:
$\underline{p}_1^{\mu} + \alpha_{p_1}(\overline{p}_1^{\mu} - \underline{p}_1^{\mu}) + i[\underline{q}_1^{\mu} + \alpha_{q_1}(\overline{q}_1^{\mu} - \underline{q}_1^{\mu})] = \underline{p}_2^{\mu} + \alpha_{p_2}(\overline{p}_2^{\mu} - \underline{p}_2^{\mu}) + i[\underline{q}_2^{\mu} + \alpha_{q_2}(\overline{q}_2^{\mu} - \underline{q}_2^{\mu})]$;
$\tilde{z}_1 = \tilde{z}_2$,
where $\mu, \alpha_{p_1}, \alpha_{p_2}, \alpha_{p_3}, \alpha_{q_1}, \alpha_{q_2}, \alpha_{q_3} \in [0, 1]$. □

ICCS Camera Ready Version 2025
To cite this paper please use the final published version:
DOI: 10.1007/978-3-031-97573-8_14

**Lemma 7.** *For any FCN $\tilde{z}_1 = \tilde{p}_1 + i\tilde{q}_1$, $\tilde{z}_2 = \tilde{p}_2 + i\tilde{q}_2$ and $\tilde{z}_3 = \tilde{p}_3 + i\tilde{q}_3$ with HMF $\mathcal{H}(\tilde{z}_1) = z_1^{gr}(\mu, \alpha_{p_1}, \alpha_{q_1})$, $\mathcal{H}(\tilde{z}_2) = z_2^{gr}(\mu, \alpha_{p_2}, \alpha_{q_2})$ and $\mathcal{H}(\tilde{z}_3) = z_3^{gr}(\mu, \alpha_{p_3}, \alpha_{q_3})$, if $0 \notin \tilde{z}_3$, the cancellation law for multiplication (8) holds in the FCN with HMF arithmetic,*

$$\tilde{z}_1\tilde{z}_3 = \tilde{z}_2\tilde{z}_3 \Rightarrow \tilde{z}_1 = \tilde{z}_2. \tag{8}$$

*Proof.* $\tilde{z}_1 \cdot \tilde{z}_3 = \tilde{z}_2 \cdot \tilde{z}_3$;
$[\underline{p}_1^\mu + \alpha_{p_1}(\overline{p}_1^\mu - \underline{p}_1^\mu) + i[\underline{q}_1^\mu + \alpha_{q_1}(\overline{q}_1^\mu - \underline{q}_1^\mu)]] \cdot [\underline{p}_3^\mu + \alpha_{p_3}(\overline{p}_3^\mu - \underline{p}_3^\mu) + i[\underline{q}_3^\mu + \alpha_{q_3}(\overline{q}_3^\mu - \underline{q}_3^\mu)]] =$
$[\underline{p}_2^\mu + \alpha_{p_2}(\overline{p}_2^\mu - \underline{p}_2^\mu) + i[\underline{q}_2^\mu + \alpha_{q_2}(\overline{q}_2^\mu - \underline{q}_2^\mu)]] \cdot [\underline{p}_3^\mu + \alpha_{p_3}(\overline{p}_3^\mu - \underline{p}_3^\mu) + i[\underline{q}_3^\mu + \alpha_{q_3}(\overline{q}_3^\mu - \underline{q}_3^\mu)]]$;
Multiplying by $1/\tilde{z}_3$ both sides, we have:
$\underline{p}_1^\mu + \alpha_{p_1}(\overline{p}_1^\mu - \underline{p}_1^\mu) + i[\underline{q}_1^\mu + \alpha_{q_1}(\overline{q}_1^\mu - \underline{q}_1^\mu)] = \underline{p}_2^\mu + \alpha_{p_2}(\overline{p}_2^\mu - \underline{p}_2^\mu) + i[\underline{q}_2^\mu + \alpha_{q_2}(\overline{q}_2^\mu - \underline{q}_2^\mu)]$;
$\tilde{z}_1 = \tilde{z}_2$,
where $\mu, \alpha_{p_1}, \alpha_{p_2}, \alpha_{p_3}, \alpha_{q_1}, \alpha_{q_2}, \alpha_{q_3} \in [0,1]$. □

The special cases of the fuzzy complex number (FCN) are triangular and trapezoidal FCNs. In the following two remarks, the triangular and trapezoidal FCNs granular approaches are defined.

*Remark 1.* (Triangular horizontal fuzzy complex membership function). A triangular fuzzy complex number $\tilde{v}^\mu = [a_1 + (b_1 - a_1)\mu, c_1 - (c_1 - b_1)\mu] + i[a_2 + (b_2 - a_2)\mu, c_2 - (c_2 - b_2)\mu]$ in the granular notation has a form of (9), where $\mu$, $\alpha_p$, $\alpha_q \in [0,1]$, see Fig. 2.

$$\begin{aligned} v^{gr}(\mu, \alpha_p, \alpha_q) = a_1 + (b_1 - a_1)\mu + [c_1 - a_1 - \mu(c_1 - a_1)]\alpha_p \\ + i[a_2 + (b_2 - a_2)\mu + [c_2 - a_2 - \mu(c_2 - a_2)]\alpha_q] \end{aligned} \tag{9}$$

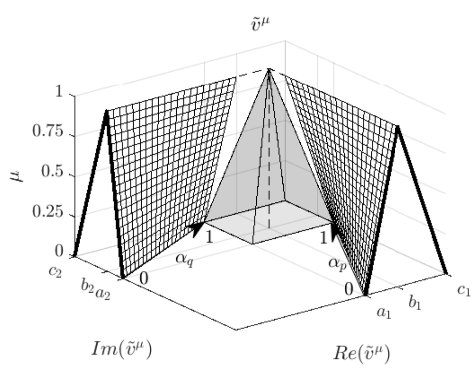


**Fig. 2.** Triangular fuzzy complex number $\tilde{v}^\mu = [a_1 + (b_1 - a_1)\mu, c_1 - (c_1 - b_1)\mu] + i[a_2 + (b_2 - a_2)\mu, c_2 - (c_2 - b_2)\mu]$, $\mu \in [0,1]$ and variables $\alpha_p, \alpha_q \in [0,1]$ generating the information granule.

ICCS Camera Ready Version 2025
To cite this paper please use the final published version:
DOI: 10.1007/978-3-031-97573-8_14

*Remark 2.* (Trapezoidal horizontal fuzzy complex membership function). A trapezoidal fuzzy complex number $\tilde{w}^{\mu} = [a_1 + (b_1 - a_1)\mu, d_1 - (d_1 - c_1)\mu] + i[a_2 + (b_2 - a_2)\mu, d_2 - (d_2 - c_2)\mu]$ in the granular notation has a form of (10), where $\mu$, $\alpha_p$, $\alpha_q \in [0, 1]$, see Fig. 3.

$$w^{gr}(\mu, \alpha_p, \alpha_q) = a_1 + (b_1 - a_1)\mu + [d_1 - a_1 - \mu(d_1 - a_1 + b_1 - c_1)]\alpha_p \\ + i[a_2 + (b_2 - a_2)\mu + [d_2 - a_2 - \mu(d_2 - a_2 + b_2 - c_2)]\alpha_q] \tag{10}$$

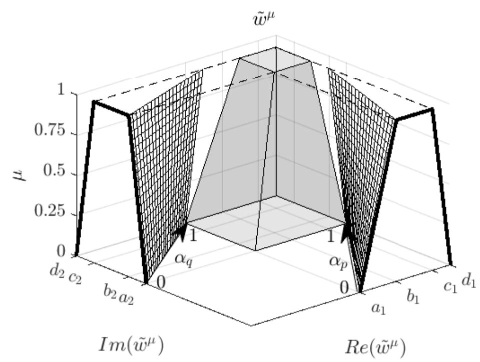


**Fig. 3.** Trapezoidal fuzzy complex number $\tilde{w}^{\mu} = [a_1 + (b_1 - a_1)\mu, d_1 - (d_1 - c_1)\mu] + i[a_2 + (b_2 - a_2)\mu, d_2 - (d_2 - c_2)\mu]$, $\mu \in [0, 1]$ and variables $\alpha_p, \alpha_q \in [0, 1]$ generating the information granule.

## 3 Examples of the FCLS Solved with the Granular Fuzzy Complex Membership Function

In this section three examples from [3, 4, 11, 14, 15, 28] using horizontal approach to fuzzy complex numbers are solved. The last example presented in the paper - Example 3 - considers the full fuzzy complex linear system and its multidimensional granular solution.

Moreover, in the presented examples it was shown that the cited papers do not provide a full solution. On the basis of the fuzzy complex linear system (FCLS), infinitely many crisp complex linear systems (CCLS) can be generated. To show that results obtained in [3, 4, 11, 13–15, 28] are not full solutions, the CCLS generated from the analyzed examples of FCLS are examined, i.e. for support when $\mu = 0$. If the methods give a full solution, then the solution of the CCLS should be included in the result of the FCLS, for the same constant $\mu = 0$.

ICCS Camera Ready Version 2025
To cite this paper please use the final published version:
DOI: 10.1007/978-3-031-97573-8_14

### 3.1 Examples

*Example 1.* (Example 1 [3, 4], example 4.1 [15], example 1 [14], example 5.1 [11]) Let us consider the fuzzy complex linear system (11).

$$\begin{aligned} \tilde{z}_1 - \tilde{z}_2 &= [\mu, 2-\mu] + i[1+\mu, 3-\mu] \\ \tilde{z}_1 + 3\tilde{z}_2 &= [4+\mu, 7-2\mu] + i[\mu-4, -1-2\mu] \end{aligned} \tag{11}$$

The FCLS (11) in granular notation has a form of (12),

$$\begin{aligned} z_1^{gr} - z_2^{gr} &= \mu + (2-2\mu)\alpha_{p_1} + i(1+\mu+(2-\mu)\alpha_{q_1}) \\ z_1^{gr} + 3z_2^{gr} &= 4+\mu+(3-3\mu)\alpha_{p_2} + i(-4+\mu+(3-3\mu)\alpha_{q_2}) \end{aligned} \tag{12}$$

where $\mu, \alpha_{p_1}, \alpha_{p_2}, \alpha_{q_1}, \alpha_{q_2} \in [0,1]$.

With basic algebraic operations the multidimensional granular solution was obtained, formula (13),

$$\begin{aligned} z_1^{gr}(\mu, \alpha_{p_1}, \alpha_{p_2}, \alpha_{q_1}, \alpha_{q_2}) &= (4+6\alpha_{p_1}+3\alpha_{p_2}+\mu(4-6\alpha_{p_1}-3\alpha_{p_2}))/4 \\ &\quad +i(-1+6\alpha_{q_1}+3\alpha_{q_2}+\mu(4-6\alpha_{q_1}-3\alpha_{q_2}))/4 \\ z_2^{gr}(\mu, \alpha_{p_1}, \alpha_{p_2}, \alpha_{q_1}, \alpha_{q_2}) &= (4-2\alpha_{p_1}+3\alpha_{p_2}+\mu(2\alpha_{p_1}-3\alpha_{p_2}))/4 \\ &\quad +i(-5-2\alpha_{q_1}+3\alpha_{q_2}+\mu(2\alpha_{q_1}-3\alpha_{q_2}))/4 \end{aligned} \tag{13}$$

where $\mu, \alpha_{p_1}, \alpha_{p_2}, \alpha_{q_1}, \alpha_{q_2} \in [0,1]$.

The granular solution (13) satisfies the FCLS (11). Substituting the obtained solution for $\tilde{z}_1$ and $\tilde{z}_2$ in the FCLS, we have:

$\tilde{z}_1 - \tilde{z}_2 = [(4+6\alpha_{p_1}+3\alpha_{p_2}+\mu(4-6\alpha_{p_1}-3\alpha_{p_2}))/4 + i(-1+6\alpha_{q_1}+3\alpha_{q_2}+\mu(4-6\alpha_{q_1}-3\alpha_{q_2}))/4] - [(4-2\alpha_{p_1}+3\alpha_{p_2}+\mu(2\alpha_{p_1}-3\alpha_{p_2}))/4 + i(-5-2\alpha_{q_1}+3\alpha_{q_2}+\mu(2\alpha_{q_1}-3\alpha_{q_2}))/4] = 2\alpha_{p_1}+\mu(1-2\alpha_{p_1}) + i(1+2\alpha_{q_1}+\mu(1-2\alpha_{q_1})) = r+(2-2\mu)\alpha_{p_1} + i(1+\mu+(2-2\mu)\alpha_{q_1}) = [\mu, 2-\mu] + i[1+\mu, 3-\mu]$;

$\tilde{z}_1 + 3\tilde{z}_2 = [(4+6\alpha_{p_1}+3\alpha_{p_2}+\mu(4-6\alpha_{p_1}-3\alpha_{p_2}))/4 + i(-1+6\alpha_{q_1}+3\alpha_{q_2}+\mu(4-6\alpha_{q_1}-3\alpha_{q_2}))/4] + 3[(4-2\alpha_{p_1}+3\alpha_{p_2}+\mu(2\alpha_{p_1}-3\alpha_{p_2}))/4 + i(-5-2\alpha_{q_1}+3\alpha_{q_2}+\mu(2\alpha_{q_1}-3\alpha_{q_2}))/4] = 4+3\alpha_{p_2}+\mu(1-3\alpha_{p_2}) + i(-4+3\alpha_{q_2}+\mu(1-3\alpha_{q_2})) = 4+\mu+(3-3\mu)\alpha_{p_2} + i(-4+\mu+(3-3\mu)\alpha_{q_2}) = [4+\mu, 7-2\mu] + i[\mu-4, -1-2\mu]$.

The granular solution also satisfies equivalent form (14) of the FCLS (11).

$$\begin{aligned} \tilde{z}_1 - [\mu, 2-\mu] - i[1+\mu, 3-\mu] &= \tilde{z}_2 \\ \tilde{z}_1 - [4+\mu, 7-2\mu] - i[\mu-4, -1-2\mu] &= -3\tilde{z}_2 \end{aligned} \tag{14}$$

Substituting multidimensional granular solution (13) into FCLS (14), we have:

$\tilde{z}_1 - [\mu, 2-\mu] - i[1+\mu, 3-\mu] = [(4+6\alpha_{p_1}+3\alpha_{p_2}+\mu(4-6\alpha_{p_1}-3\alpha_{p_2}))/4 + i(-1+6\alpha_{q_1}+3\alpha_{q_2}+\mu(4-6\alpha_{q_1}-3\alpha_{q_2}))/4] - [\mu+(2-2\mu)\alpha_{p_1} + i(1+\mu+(2-2\mu)\alpha_{q_1})] = (4-2\alpha_{p_1}+3\alpha_{p_2}+\mu(2\alpha_{p_1}-3\alpha_{p_2}))/4 + i(-5-2\alpha_{q_1}+3\alpha_{q_2}+\mu(2\alpha_{q_1}-3\alpha_{q_2}))/4 = \tilde{z}_2$;

$\tilde{z}_1 - [4+\mu, 7-2\mu] - i[\mu-4, -1-2\mu] = [(4+6\alpha_{p_1}+3\alpha_{p_2}+\mu(4-6\alpha_{p_1}-3\alpha_{p_2}))/4 + i(-1+6\alpha_{q_1}+3\alpha_{q_2}+\mu(4-6\alpha_{q_1}-3\alpha_{q_2}))/4] - [4+\mu+(3-3\mu)\alpha_{p_2} + i(-4+\mu+(3-3\mu)\alpha_{q_2})] = (-12+6\alpha_{p_1}-9\alpha_{p_2}-6\mu\alpha_{p_1}+9\mu\alpha_{p_2})/4 + i(15+6\alpha_{q_1}-9\alpha_{q_2}-6\mu\alpha_{q_1}+9\mu\alpha_{q_2})/4 = -3[(4-2\alpha_{p_1}+3\alpha_{p_2}+\mu(2\alpha_{p_1}-3\alpha_{p_2}))/4 + i(-5-2\alpha_{q_1}+3\alpha_{q_2}+\mu(2\alpha_{q_1}-3\alpha_{q_2}))/4] = -3\tilde{z}_2$.

ICCS Camera Ready Version 2025
To cite this paper please use the final published version:
DOI: 10.1007/978-3-031-97573-8_14

Similarly, it can be shown that the obtained granular solution satisfies other equivalent forms of the FCLS.

The span of the granular solution (13) in the form of fuzzy complex numbers presents (15) and graphically in Fig. 4,

$$\begin{aligned} \tilde{z}_1^{\mu} = \mathcal{H}^{-1}(z_1^{gr}(\mu, \alpha_{p_1}, \alpha_{p_2}, \alpha_{q_1}, \alpha_{q_2})) &= [1+\mu, 3.25-1.25\mu] \\ &\quad +i[-0.25+\mu, 2-1.25\mu], \\ \tilde{z}_2^{\mu} = \mathcal{H}^{-1}(z_2^{gr}(\mu, \alpha_{p_1}, \alpha_{p_2}, \alpha_{q_1}, \alpha_{q_2})) &= [0.5+0.5\mu, 1.75-0.75\mu] \\ &\quad +i[-1.75+0.5\mu, -0.5-0.75\mu], \end{aligned} \tag{15}$$

where $\mu \in [0, 1]$.

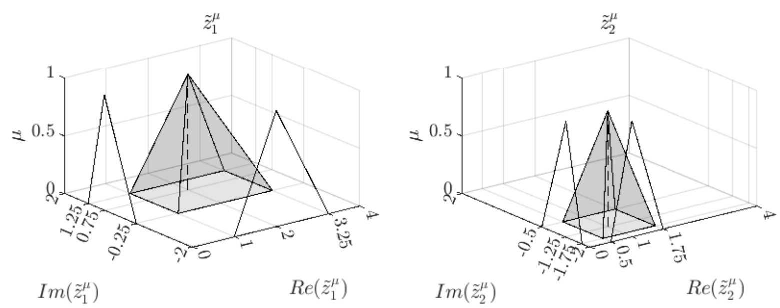


**Fig. 4.** Spans of complex granule of solution of fuzzy complex system of linear equations, Example 1.

The next Example 2 taken from [28, 11] considers a simple RLC circuit where both current and source are fuzzy.

*Example 2.* (Example 2 [28], example 5.5 [11]) Let us consider the FCLS (16) for RLC circuit presented in Fig. 5.

$$\begin{aligned} (10-7.5i)\tilde{z}_1 + (-6+5i)\tilde{z}_2 &= [4+\mu, 6-\mu] + i[-1+\mu, 1-\mu] \\ (-6+5i)\tilde{z}_1 + (16+3i)\tilde{z}_2 &= [-2+\mu, -\mu] + i[-3+\mu, -1-\mu] \end{aligned} \tag{16}$$

where $\mu \in [0, 1]$.

FCLS (16) in the granular notation has a form (17)

$$\begin{aligned} (10-7.5i)z_1^{gr} + (-6+5i)z_2^{gr} &= 4+\mu+\alpha_{p_1}(2-2\mu) + i[-1+\mu+\alpha_{q_1}(2-2\mu)] \\ (-6+5i)z_1^{gr} + (16+3i)z_2^{gr} &= -2+\mu+\alpha_{p_2}(2-2\mu) + i[-3+\mu+\alpha_{q_2}(2-2\mu)] \end{aligned} \tag{17}$$

where $\mu, \alpha_{p_1}, \alpha_{p_2}, \alpha_{q_1}, \alpha_{q_2} \in [0, 1]$.

ICCS Camera Ready Version 2025
To cite this paper please use the final published version:
DOI: 10.1007/978-3-031-97573-8_14

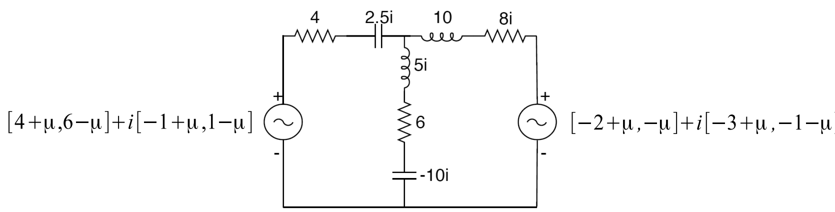


**Fig. 5.** A simple RLC circuit where both current and source are fuzzy [28, 11].

The multidimensional granular form of solution is presented by (18).

$$\begin{aligned} & z_1^{gr}\ (\mu,\alpha_{p_1},\alpha_{p_2},\alpha_{q_1},\alpha_{q_2}) \\ & =\{21232\alpha_{p_1}+9432\alpha_{p_2}-7956\alpha_{q_1}+5420\alpha_{q_2}+28880 \\ & \quad +\mu(14064-21232\alpha_{p_1}-9432\alpha_{p_2}+7956\alpha_{q_1}-5420\alpha_{q_2}) \\ & \quad +i[7956\alpha_{p_1}-5420\alpha_{p_2}+21232\alpha_{q_1}+9432\alpha_{q_2}-3432 \\ & \quad +\mu(16600-7956\alpha_{p_1}+5420\alpha_{p_2}-21232\alpha_{q_1}-9432\alpha_{q_2})]\} \\ & \quad /121249 \\ & z_2^{gr}\ (\mu,\alpha_{p_1},\alpha_{p_2},\alpha_{q_1},\alpha_{q_2}) \\ & =\{9432\alpha_{p_1}+15520\alpha_{p_2}+5420\alpha_{q_1}+7890\alpha_{q_2}-11201 \\ & \quad +\mu(19131-9432\alpha_{p_1}-15520\alpha_{p_2}-5420\alpha_{q_1}-7890\alpha_{q_2}) \\ & \quad +i[9432\alpha_{p_1}-7890\alpha_{p_2}-5420\alpha_{q_1}+15520\alpha_{q_2}-30946 \\ & \quad +\mu(5821+5420\alpha_{p_1}+7890\alpha_{p_2}-9432\alpha_{q_1}-15520\alpha_{q_2})]\} \\ & \quad /121249 \end{aligned} \tag{18}$$

Obtained multidimensional solution (18) satisfies the considered FCLS and its equivalent forms. It can be proved similarly as in Example 1.

The span of the multidimensional granular solution expressed in the form of the FCN is presented by formula (19) and in Fig. 6

$$\begin{aligned} \tilde{z}_1^{\mu} &= \mathcal{H}^{-1}(z_1^{gr}(\mu,\alpha_{p_1},\alpha_{p_2},\alpha_{q_1},\alpha_{q_2})) = [0.1726+0.1816\mu, 0.5358-0.1816\mu] \\ &\qquad +i[-0.0730+0.1816\mu, 0.2902-0.1816\mu] \\ \tilde{z}_2^{\mu} &= \mathcal{H}^{-1}(z_2^{gr}(\mu,\alpha_{p_1},\alpha_{p_2},\alpha_{q_1},\alpha_{q_2})) = [-0.0924+0.1578\mu, 0.2232-0.1578\mu] \\ &\qquad +i[-0.3650+0.1578\mu, -0.0494-0.1578\mu] \end{aligned} \tag{19}$$

*Example 3.* Let us consider the full fuzzy complex linear system (20).

$$\begin{aligned} ([\mu,2-\mu]+i[\mu,2-\mu])\tilde{z}_1 &+ ([-3+\mu,-1-\mu]+i[5+\mu,7-\mu])\tilde{z}_2 \\ &= [-11+\mu,-9-\mu]+i[11+\mu,13-\mu], \\ ([-8+\mu,-6-\mu]+i[-5+\mu,-3-\mu])\tilde{z}_1 &+ ([\mu,2-\mu]+i[\mu,2-\mu])\tilde{z}_2 \\ &= [-3+\mu,-1-\mu]+i[-9+\mu,-7-\mu]. \end{aligned} \tag{20}$$

ICCS Camera Ready Version 2025
To cite this paper please use the final published version:
DOI: 10.1007/978-3-031-97573-8_14

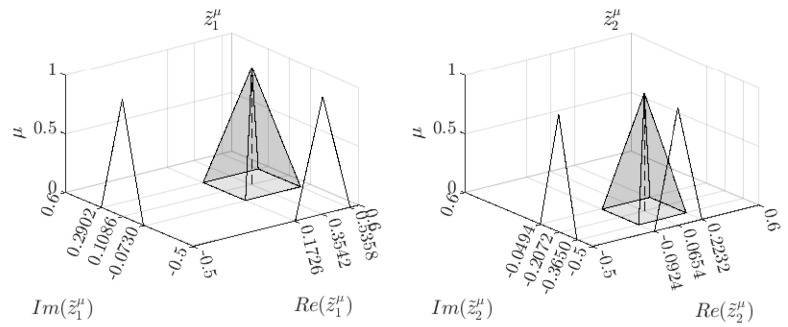


**Fig. 6.** Span of the multidimensional granule of solution $\tilde{z}_1^\mu$ and $\tilde{z}_2^\mu$ from Example 2.

Fuzzy complex linear system (20) in multidimensional granular notation has a form of (21),

$$\begin{aligned}
&[\mu + \alpha_{p_1}(2 - 2\mu) + i[\mu + \alpha_{q_1}(2 - 2\mu)]]z_1^{gr} \\
&\qquad + [-3 + \mu + \alpha_{p_2}(2 - 2\mu) + i[5 + \mu + \alpha_{q_2}(2 - 2\mu)]]z_2^{gr} \\
&\qquad = -11 + \mu + \alpha_{p_3}(2 - 2\mu) + i[-5 + \mu + \alpha_{q_3}(2 - 2\mu)], \\
&[-8 + \mu + \alpha_{p_4}(2 - 2\mu) + i[-5 + \mu + \alpha_{q_4}(2 - 2\mu)]]z_1^{gr} \\
&\qquad + [\mu + \alpha_{p_5}(2 - 2\mu) + i[\mu + \alpha_{q_5}(2 - 2\mu)]]z_2^{gr} \\
&\qquad = -3 + \mu + \alpha_{p_6}(2 - 2\mu) + i[-9 + \mu + \alpha_{q_6}(2 - 2\mu)],
\end{aligned} \tag{21}$$

where $\mu, \alpha_{p_1}, \ldots, \alpha_{p_6}, \alpha_{q_1}, \ldots, \alpha_{q_6} \in [0, 1]$.

The granule of determinant $d^{gr}(\mu, \alpha_p, \alpha_q)$ of the coefficients of uncertain values of the FCLS (21) is equal as in equation (22),

$$\begin{aligned}
d^{gr}(\mu, \alpha_p, \alpha_q) = &[\mu + \alpha_{p_1}(2 - 2\mu) + i[\mu + \alpha_{q_1}(2 - 2\mu)]] \\
&\cdot[\mu + \alpha_{p_5}(2 - 2\mu) + i[\mu + \alpha_{q_5}(2 - 2\mu)]] \\
&-[-3 + \mu + \alpha_{p_2}(2 - 2\mu) + i[5 + \mu + \alpha_{q_2}(2 - 2\mu)]] \\
&\cdot[-8 + \mu + \alpha_{p_4}(2 - 2\mu) + i[-5 + \mu + \alpha_{q_4}(2 - 2\mu)]],
\end{aligned} \tag{22}$$

where $\alpha_p = \{\alpha_{p_1}, \ldots, \alpha_{p_6}\}$, $\alpha_q = \{\alpha_{q_1}, \ldots, \alpha_{q_6}\}$ and $\mu$, $\alpha_{p_1}$, ..., $\alpha_{p_6}$, $\alpha_{q_1}$, ..., $\alpha_{q_6} \in [0, 1]$.

Equation (23) and Fig. 7 present the span of the determinant $d^{gr}(\mu, \alpha_p, \alpha_q)$ for border values of the variables $\alpha_p = \{\alpha_{p_1}, \ldots, \alpha_{p_6}\}$, $\alpha_q = \{\alpha_{q_1}, \ldots, \alpha_{q_6}\}$, $0 \notin d^{gr}(\mu, \alpha_p, \alpha_q)$.

$$\tilde{d}^\mu = \mathcal{H}^{-1}(d^{gr}(\mu, \alpha_p, \alpha_q)) = [-63 + 25\mu, -17 - 21\mu] + i[15 + 21\mu, 61 - 25\mu], \tag{23}$$

where $\mu \in [0, 1]$.

ICCS Camera Ready Version 2025
To cite this paper please use the final published version:
DOI: 10.1007/978-3-031-97573-8_14

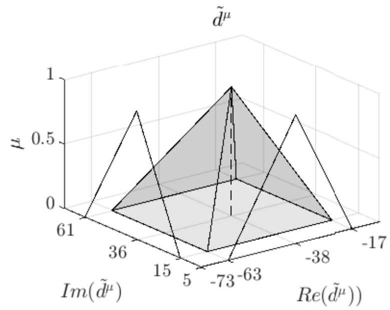


**Fig. 7.** Span of determinant $\tilde{d}^{\mu}$.

With basic algebraic operations, the multidimensional granular solution of the FCLS (21) equals as presented with equation (24)

$$\begin{aligned}
z_1^{gr}\ (\mu,\alpha_p,\alpha_q) &= \{[-11+\mu+\alpha_{p_3}(2-2\mu)+i[-5+\mu+\alpha_{q_3}(2-2\mu)]] \\
&\cdot[\mu+\alpha_{p_5}(2-2\mu)+i[\mu+\alpha_{q_5}(2-2\mu)]]-[-3+\mu+\alpha_{p_2}(2-2\mu) \\
&+i[5+\mu+\alpha_{q_2}(2-2\mu)]]\cdot[-3+\mu+\alpha_{p_6}(2-2\mu) \\
&+i[-9+\mu+\alpha_{q_6}(2-2\mu)]]\}/d^{gr}(\mu,\alpha_p,\alpha_q)) \\
z_2^{gr}\ (\mu,\alpha_p,\alpha_q) &= \{[\mu+\alpha_{p_1}(2-2\mu)+i[\mu+\alpha_{q_1}(2-2\mu)]] \\
&\cdot[-3+\mu+\alpha_{p_6}(2-2\mu)+i[-9+\mu+\alpha_{q_6}(2-2\mu)]] \\
&-[-11+\mu+\alpha_{p_3}(2-2\mu)+i[-5+\mu+\alpha_{q_3}(2-2\mu)]] \\
&\cdot[-8+\mu+\alpha_{p_4}(2-2\mu)+i[-5+\mu+\alpha_{q_4}(2-2\mu)]]\}/d^{gr}(\mu,\alpha_p,\alpha_q))
\end{aligned} \tag{24}$$

where $\alpha_p = \{\alpha_{p_1},\ldots,\alpha_{p_6}\}$, $\alpha_q = \{\alpha_{q_1},\ldots,\alpha_{q_6}\}$ and $\mu$, $\alpha_{p_1}$, …, $\alpha_{p_6}$, $\alpha_{q_1},\ldots,\alpha_{q_6}$ $\in [0,1]$.

To prove that the obtained solution satisfies the FCLS and its equivalent forms, the granular solution should be used in the place of variables in the system, similarly as in the Example 1.

The span of the granular solution (24) is presented in Fig. 8. The spans in Fig. 8 were calculated for border values $\{0,1\}$ of the horizontal variables and $\mu \in [0:0.2:1]$. In this example the left and the right sides of the spans of the solution are nonlinear.

Approximations of the spans by the triangular fuzzy complex number using the supports and the cores of the spans are presented with the equation (25),

$$\begin{aligned}
\tilde{z}_1^{\mu} = \mathcal{H}^{-1}(z_1^{gr}(\mu,\alpha_p,\alpha_q)) &= [0.1584+0.8416\mu, 2.0902-1.0902\mu] \\
&+i[0.0338+0.9662\mu, 2.3312-1.3312\mu], \\
\tilde{z}_2^{\mu} = \mathcal{H}^{-1}(z_2^{gr}(\mu,\alpha_p,\alpha_q)) &= [0.9292+1.0708\mu, 3.1375-1.1375\mu] \\
&+i[-0.1315+1.1315\mu, 2.5344-1.5344\mu],
\end{aligned} \tag{25}$$

where $\mu \in [0,1]$.

ICCS Camera Ready Version 2025
To cite this paper please use the final published version:
DOI: 10.1007/978-3-031-97573-8_14

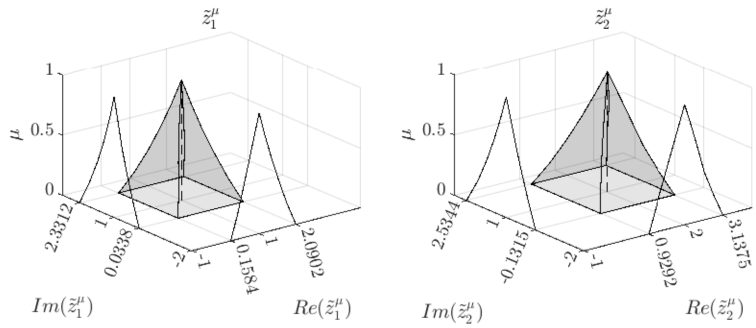


**Fig. 8.** The span of the multidimensional granular solution from Example 3.

### 3.2 Comparison of Results

Let us consider the solution obtained in Example 1. The fuzzy complex numbers (15) are different than the results with the method provided by the authors in [3, 4, 11, 14, 15].

From the FCLS in example 1 [3, 4], example 4.1 [15], example 5.1 [11] and example 1 [14], for $\mu = 0$ the CCLS (26) can be obtained:

$$\begin{aligned} z_1 - z_2 &= 2 + 3i, \\ z_1 + 3z_2 &= 7 - i, \end{aligned} \tag{26}$$

where the crisp solution equals: $z_1 = 3.25 + 2i$ and $z_2 = 1.25 - i$. Unfortunately, the crisp solution $z_1$ does not belong to the solution of the FCLS presented in ([3, 4, 11, 14, 15]), for $\mu = 0$ we have: $z_1 = 3.25 + 2i \notin \tilde{z}_1(\mu = 0) = [1.375, 2.875] + i[0.125, 1.625]$. It shows that results presented in ([3, 4, 14, 15]) are not a full solution.

With the multidimensional granular solution (13) using the horizontal variables all pairs of the crisp complex solutions can be generated. For example, for $\mu = 0$ and $\alpha_{p_1} = \alpha_{p_2} = \alpha_{q_1} = \alpha_{q_2} = 1$ the crisp solution of the crisp complex linear system (26) can be obtained.

Let us analyze solution of Example 2. The solution (18) obtained with multidimensional horizontal approach is different than results calculated in papers [11, 28]. Let us consider the CCLS (27) from FCLS (16), where $\mu = 0$

$$\begin{aligned} (10 - 7.5i)z_1 + (-6 + 5i)z_2 &= 4 + i \\ (-6 + 5i)z_1 + (16 + 3i)z_2 &= -2 - 3i \end{aligned} \tag{27}$$

Crisp solution of the CCLS (27) equals: $z_1 = 0.1726 + 0.1468i$, $z_2 = -0.0477 - 0.1774i$. Value $z_1$ does not belong to the results in [28, 11], for $\mu = 0$ we have: $z_1 = 0.1726 + 0.1468i \notin \tilde{z}_1(\mu = 0) = [0.3164, 0.3920] + i[0.0708, 0, 1464]$. So the results in [11, 28] are not full solutions.

The crisp solution of CCLS (27) belongs to the direct solution with HFCN (18), it can be obtained for $\mu = \alpha_{p_1} = \alpha_{p_2} = \alpha_{q_2} = 0$ and $\alpha_{q_1} = 1$.

ICCS Camera Ready Version 2025
To cite this paper please use the final published version:
DOI: 10.1007/978-3-031-97573-8_14

## 4 Conclusions

The paper presents a multidimensional approach to solving fuzzy systems of linear equations using fuzzy complex numbers (FCN) and their horizontal membership functions (HMF). It is also proved that defined operations on FCNs with HMFs hold basic algebraic properties. With arithmetic on the fuzzy complex numbers with HMFs the fuzzy complex linear systems (FCLS) were solved. Obtained solution is a granule of information expressed in the form of the formula. What is more, it was shown that the multidimensional solution satisfies the FCLS and its equivalent forms. With a multidimensional solution it is possible to generate any crisp solution. This feature helps to prove that presented results obtained in the cited papers [3, 4, 11, 15, 28] are not full solutions of the FCLS. With a multidimensional solution, a crisp solution of the FCLS was generated, which is not present in the results obtained with the methods used in cited papers.

Presented approach of the complex horizontal fuzzy membership function and their arithmetic can be used in many problems of granular computations.

## References


1. Abbasbandy, S., Alavi, M.: A method for solving fuzzy linear systems. Iran. J. Fuzzy Syst. **2**(2), 37–43 (2005)
2. Behera, D., Chakraverty, S..: A new method for solving real and complex fuzzy systems of linear equations. Comput. Math. Model. **23**(4), 507–518 (2012)
3. Behera, D., Chakraverty, S.: Solving fuzzy complex system of linear equations. Inf. Sci. **277**, 154–162 (2014)
4. Behera, D., Chakraverty, S.: Erratum to "Solving fuzzy complex system of linear equations" [Information sciences 277 (2014) 154–162]. Inf. Sci. **369**, 788–790 (2016)
5. Boroujeni, M., Basiri, A., Rahmany, S., Valibouze, A.: Solving fuzzy systems in dual form using wu's method. Int. J. Fuzzy Syst. **17**(2), 170–180 (2015)
6. Buckley, J.J.: Fuzzy complex numbers. Fuzzy Sets Syst. **33**(3), 333–345 (1989)
7. Buckley, J.J.: Fuzzy complex analysis II: integration. Fuzzy Sets Syst. **49**(2), 171–179 (1992)
8. Chang, S.S.L., Zadeh, L.A.: On fuzzy mapping and control. IEEE Trans. Systems Man. Cybernet. **2**, 30–34 (1972)
9. Dehghan, M., Hashemiand, B., Ghatee, M.: Computational methods for solving fully fuzzy linear systems. Appl. Math. Comput. **179**, 328–343 (2006)
10. Dubois, D., Prade, H.: Operations on fuzzy numbers. Int. J. Systems Sci. **9**, 613–626 (1978)
11. Farahani, H., Nehi, H.M., Paripour, M.: Solving fuzzy complex system of linear equations using eigenvalue method. J. Intell. Fuzzy Syst. **31**(3), 1689–1699 (2016)
12. Friedman, M., Ming, M., Kandel, A.: Fuzzy linear systems. Fuzzy Sets Syst. **96**, 201–209 (1998)
13. Ghanbari, M.: A discussion on "solving fuzzy complex system of linear equations". Inf. Sci. **402**, 165–169 (2017)
14. Guo, X., Zhang, K.: Minimal Solution of Complex Fuzzy Linear Systems. Adv. Fuzzy Syst. **9**, 1–9 (2016)


ICCS Camera Ready Version 2025
To cite this paper please use the final published version:
DOI: 10.1007/978-3-031-97573-8_14

15. Han, Y., Guo, X.: Complex fuzzy linear systems. Int. J. Eng. Appl. Sci. **3**, 30–34 (2016)
16. Jafari, H., Saeidy, M., Vahidi, J.: The homotopy analysis method for solving fuzzy system of linear equations. Int. J. Fuzzy Syst **11**(4), 308–313 (2009)
17. Jafarian, A., Nia, S.: On the solution of nonlinear fuzzy equation systems by fuzzy neural network method. Int. J. Fuzzy Syst. **15**(3), 376–380 (2013)
18. Jahantigh, M.A., Khezerloo, S., Khezerloo, M.: Complex fuzzy linear systems. Int. J. Industrial Mathematics **2**, 21–28 (2010)
19. Kaufmann, A., Gupta, M.M.: Introduction to fuzzy arithmetic: theory and applications. Van Nostrand Reinhold Co., New York, N.Y. (1985)
20. Klir, G.J., Yuan, B.: Fuzzy sets and fuzzy logic: Theory and applications. Prentice Hall, Upper Saddle River, NJ (1995)
21. Mazandarani, M., Pariz, N., Kamyad, A.V.: Granular Differentiability of Fuzzy-Number-Valued Functions. IEEE Trans. Fuzzy Syst. **26**(1), 310–323 (2018)
22. Mizumoto, M., Tanaka, K.: The four operations of arithmetic on fuzzy numbers. Systems Comput. Controls **7**, 73–81 (1976)
23. Najariyan, M., Zhao, Y.: On the stability of fuzzy linear dynamical systems. Journal of the Franklin Institute **357**, 5502–5522 (2020)
24. Piegat, A., Landowski, M.: Two interpretations of multidimensional RDM interval arithmetic - multiplication and division. Int. J. Fuzzy Syst. **16**, 488–496 (2013)
25. Piegat, A., Landowski, M.: Horizontal membership function and examples of its applications. Int. J. Fuzzy Syst. **17**, 22–30 (2015)
26. Piegat, A., Landowski, M.: Is an interval the right result of arithmetic operations on intervals? Int. J. Appl. Math. Comput. Sci. **27**(3), 575–590 (2017)
27. Piegat, A., Landowski, M.: On Fuzzy RDM-Arithmetic. In: Kobayashi, Sy., Piegat, A., Pejaś, J., El Fray, I., Kacprzyk, J. (eds.) Hard and Soft Computing for Artificial Intelligence, Multimedia and Security. ACS 2016. Advances in Intelligent Systems and Computing, vol. 534, pp. 3–16. Springer, Cham (2017). https://doi.org/10.1007/978-3-319-48429-7_1
28. Rahgooy, T., Yazdi, H.S., Monsefi, R.: Fuzzy complex system of linear equations applied to circuit analysis. Int. J. Comput. Electr. Eng. **1**, 1793–8163 (2009)
29. Ramot, D., Milo, R., Friedman, M., Kandel, A.: Complex fuzzy sets. IEEE Trans. Fuzzy Syst **10**, 171–186 (2002)
30. Ramot, D., Friedman, M., Langholzn, G., Kandel, A.: Complex fuzzy logic. IEEE Trans. Fuzzy Syst **11**, 450–461 (2003)
31. Zadeh, L.A.: Fuzzy sets. Inf. Control. **8**, 338–353 (1965)
32. Zimmermann, H.J.: Fuzzy Set Theory - and Its Applications. Springer Science+Business Media New York (1985)

ICCS Camera Ready Version 2025
To cite this paper please use the final published version:
DOI: 10.1007/978-3-031-97573-8_14