A Thermodynamically Consistent Model for Compressible Fluid Flow in Fractured Porous Elastic Media

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Abstract. This paper investigates the fluid-solid coupling problem in fractured porous elastic media. The geometry of the fractures is considered a potentially non-planar interface. The model equations are of mixed-dimensional type, where the flow equations on the $d-1$ dimensional fracture surfaces are coupled with the d dimensional porous matrix. This paper considers a strongly compressible fluid flow model, where the density is chosen as the primary variable, in contrast to the slightly compressible model discussed by Girault et al[5]. (Girault et al. Mathematical Models and Methods in Applied Sciences. Vol. 25, No. 4 (2015) 587645), which takes pressure as the primary variable. We derive a thermodynamically consistent mathematical model and present its weak formulation. Energy stability is established for continuous and semi-discrete (in time) cases. The proposed model and numerical framework provide a solid foundation for simulating strongly compressible flows while maintaining thermodynamic consistency and stability.

Keywords: Poroelasticity · Fractured media · Thermodynamically Consistent model · Energy Stability Analysis.

1 Introduction

In the fields of engineering and science, the coupled flow and geomechanics problem is of significant importance in various applications, especially in hydraulic fracturing, CO_2 injection and storage, sand production, and wellbore stability prediction[7,13]. The successful resolution of these problems often depends on effective management of subsurface mechanical stability. In fractured media, the coupling of flow and geomechanics is particularly critical, as fractures are not only regions of mechanical instability but also have a significant impact on the flow profile. When fluid is injected underground, changes in fluid pressure alter the in-situ stress conditions, which in turn cause changes in the porosity of the matrix skeleton. This flow-geomechanics coupling is typically described by the Biot equation under linear conditions, which captures the elastic effects of

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the porous skeleton through a linear elastic model and incorporates fluid pressure into the stress tensor.

The presence of fractures adds complexity to the problem. We adopt the so-called mixed-dimensional model to describe the influence of fractures, where fractures are treated as two-dimensional surfaces embedded within a three-dimensional porous matrix. The flow model on the fracture is defined by Darcy's law on the fracture surface. This model is intuitively meaningful because fractures are typically thin and long structures, so the details of lateral flow are less important, with the primary focus on the tangential flow along the fracture surface. This mixed-dimensional model couples multiple physical processes across domains of different dimensions, resulting in a mathematical model that couples the Biot equation in the porous matrix with Darcys law on the fracture surface through appropriate interface conditions.

For the fracture model, Girault et al. discussed the flow model of slightly compressible fluids, the fluid compressibility c_f is assumed to be small (e.g. of the order of 10^{-8} or 10^{-9}), where the fluid density is set to a very small value. As a result, pressure was chosen as the primary variable, and numerical approximations of the fluid flow in both the matrix and fractures were made using continuous finite element and mixed finite element methods[5]. T. Almani and K. Kumar, for the same model, performed simulations in 2022 using single-rate and multirate undrained split iterative algorithms[2]. In 2024, they applied a multirate fixed stress split iterative scheme to solve a fractured Biot model[3].

The second law of thermodynamics, a fundamental principle that governs numerous physical processes, plays a critical role in the development of reliable and comprehensive mathematical models across a wide range of scientific and engineering applications[8,11]. This law generally asserts that the entropy of an isolated system will increase over time. In the case of a specific isothermal process, it leads to an energy dissipation law, which states that the total free energy will decrease over time[10]. The energy dissipation law can also be interpreted as a means of assessing or controlling stability in mathematical systems, a critical component of the well-posedness theory of mathematical models. Given its significance, ensuring thermodynamic consistency, i.e., adherence to the second law of thermodynamics, has become an increasingly important focus in the area of porous media simulation.

This paper discusses a fractured Biot model for strongly compressible fluids, where the fluid pressure is no longer a small constant. Therefore, starting from the mass conservation equation for the fluid, we do not make physical approximations to the model but directly treat the density as the primary variable. The unknowns in our model are the displacement, density, and the leakage term connecting the flow in the matrix to the flow in the fracture. We derive a thermodynamically consistent mathematical model and present its variational form. The energy conservation property of the model is proven for both the continuous and time-semi-discrete cases.

The paper is organized as follows: In Section 2, we introduce the domain setup and the mathematical formulation of the compressible fluid-solid coupling

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model with fracture. In Section 3, we begin by defining the spatial domain and its corresponding norm, and derive the variational formulation. In Section 4, for the continuous form of the model, we define the total free energy and demonstrate that it satisfies the energy dissipation law. In Section 5, we discretize the model in time using the fully implicit backward Euler scheme and prove that the time discretization satisfies energy stability.

2 Mathematical Model

2.1 Domain

We consider a fractured porous medium $\Omega \in \mathbb{R}^d$, where $d = 2$ or $d = 3$, which is linear, elastic, and isotropic. The medium is assumed to be saturated with a compressible single-phase fluid. The fractures are treated as non-planar interfaces, denoted by $\mathcal{C}$. To simplify the modeling process and avoid dealing with curved elements, we assume that both $\partial\Omega$ and the fracture $\mathcal{C}$ are polygonal surfaces. A diagram of our domain is shown in Figure 1.

In our analysis, although the fracture does not propagate (i.e., the crack front remains stationary), the fracture width can still change over time due to fluid injection into the crack and fluid leakage out of the crack into the surrounding medium. We assume that the fracture width is small enough (compared to other relevant length scales associated with the fracture) to allow the use of Reynolds lubrication equation for modeling the flow within the fracture (see [15,14,6]).

To simplify, we do not specify the time dependence of the spaces, and we denote the scalar products in space using parentheses. If the domain of integration is not specified, it is understood that the integrals are taken over the domain $\Omega \setminus \mathcal{C}$.

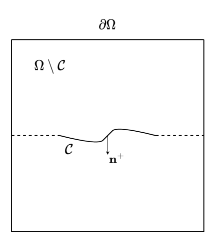

Fig. 1: Diagram of domain, fracture, and boundaries.

2.2 Model

For coupling flow with mechanics, a quasi-static Biot model is assumed, where the second-order time derivative for the displacement is ignored. The modeling

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equation system consists of

$$-\nabla \cdot \sigma^{\mathrm{por}}(\mathbf{w}, p) = \rho \mathbf{g}, \ \sigma^{\mathrm{por}}(\mathbf{w}, p) = \sigma(\mathbf{w}) - \alpha p \mathbf{I}, \quad \forall \mathbf{x} \in \Omega \setminus \mathcal{C}, \tag{1}$$

$$\sigma(\mathbf{w}) = \lambda(\nabla \cdot \mathbf{w})\mathbf{I} + 2G\epsilon(\mathbf{w}), \ \epsilon(\mathbf{w}) := \frac{1}{2}(\nabla \mathbf{w} + \nabla \mathbf{w}^T), \quad \forall \mathbf{x} \in \Omega \setminus \mathcal{C}, \tag{2}$$

$$\sigma^{\mathrm{por}} \cdot \mathbf{n} = -p\mathbf{n}, \quad \forall \mathbf{x} \in \mathcal{C}, \tag{3}$$

where σ^{por} is the Cauchy stress tensor, $\mathbf{I}$ is the identity tensor, $\mathbf{w}$ is the solids displacement, p is the fluid pressure, $\sigma(\mathbf{w})$ is the effective linear elastic stress tensor. Here $\lambda > 0$ and $G > 0$ are the Lamé constants and $\alpha > 0$ is the dimensionless Biot coefficient, $\rho \mathbf{g}$ is gravity loading term, let $\mathbf{n}$ denote the unit normal vector to $\mathcal{C}$ exterior to Ω. Let τ_j, $1 \le j \le d-1$, be a set of orthonormal tangent vectors on $\mathcal{C}$. The balance of the normal traction vector and the conservation of mass yield the interface conditions on each side (or face) of $\mathcal{C}$:

$$\sigma^{\mathrm{por}} \cdot \mathbf{n} = -p\mathbf{n}. \tag{4}$$

Then the continuity of p through $\mathcal{C}$ yields $[\sigma^{\mathrm{por}}]_{\mathcal{C}} \cdot \mathbf{n} = \mathbf{0}$. Formula (4) also implies

$$\sigma^{\mathrm{por}} \cdot \mathbf{n} \cdot \mathbf{n} = -p, \quad \sigma^{\mathrm{por}} \cdot \mathbf{n} \cdot \tau = \mathbf{0}. \tag{5}$$

For the fluid, we use a compressible single-phase model. The fluid mass balance in $\Omega \setminus \mathcal{C}$ reads

$$\frac{\partial\left((\alpha \nabla \cdot \mathbf{w} + \phi)\rho_f\right)}{\partial t} + \nabla \cdot (\rho_f \mathbf{u}) = q, \quad \forall \mathbf{x} \in \Omega \setminus \mathcal{C}, \tag{6}$$

$$\mathbf{u} = -\frac{\mathbf{K}}{\eta}(\nabla p - \rho_f \mathbf{g}), \quad \forall \mathbf{x} \in \Omega \setminus \mathcal{C}, \tag{7}$$

where ρ_f is the fluid density, $\mathbf{u}$ is the velocity of the fluid, $\mathbf{K}$ is the permeability tensor in the matrix, assumed to be symmetric, bounded, uniformly positive definite in space and constant in time, $\eta > 0$ is the constant fluid viscosity, q is a mass source or sink term taking into account injection into or out of the reservoir. We assume the porosity ϕ relies on space, but is independent of time.

The conservation of mass in the fracture reads

$$\frac{\partial\left(w_{\mathrm{frac}}\rho_f\right)}{\partial t} + \nabla_{\mathcal{C}} \cdot (\rho_f \mathbf{u}^{\mathrm{frac}}) = q^{\mathrm{frac}} - \rho_f z, \quad \forall \mathbf{x} \in \mathcal{C}, \tag{8}$$

$$w_{\mathrm{frac}} = -\left[\mathbf{w}\right]_{\mathcal{C}} \cdot \mathbf{n}^+, \ \mathbf{u}^{\mathrm{frac}} = -\frac{\mathbf{K}_{\mathrm{frac}}}{\eta}(\nabla p - \rho_f \mathbf{g}), \ z = -\left[\mathbf{u}\right]_{\mathcal{C}} \cdot \mathbf{n}^+, \tag{9}$$

where w_{frac} represents the width of the fracture, $\nabla_{\mathcal{C}}$ is the tangential derivative along the fracture, $\mathbf{u}^{\mathrm{frac}}$ is the flux unknowns in the fracture, $\mathbf{K}_{\mathrm{frac}}$ is the permeability tensor in the fracture, q^{frac} is a known injection term into the fracture. $z = -\rho_f \left[\mathbf{u}\right]_{\mathcal{C}} \cdot \mathbf{n}^+$ is the leakage term connecting the flow in the matrix to the flow in the fracture. We assume that w_{frac} is bounded in $\mathcal{C}$ and vanishes on $\partial\mathcal{C}$.

For any function f defined in $\Omega \setminus \mathcal{C}$ that has a trace, let f^* denote the trace of f on $\mathcal{C}$. Then we define the jump of f on $\mathcal{C}$ in the direction of $\mathbf{n}^+$ by

$$[f]_{\mathcal{C}} = f^+ - f^-.$$

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The total fluid compressibility by

$$\frac{\mathrm{d}\rho_f}{\mathrm{d}p} = \rho_f c_f,\ \rho = \rho_R(1-\phi) + \phi\rho_f, \quad \forall \mathbf{x} \in \Omega,$$

where the compressibility $c_f = c_f(p)$ can be obtained from the fluid density $\rho_f = \rho_f(p)$, ρ_R is the density of the solids.

Summarizing, the equations in $\Omega \setminus \mathcal{C}$ are (1) and (6), and the equation in $\mathcal{C}$ is (8); the corresponding unknowns are $\mathbf{w}$, ρ_f, and z. These equations are complemented in the next section by boundary and initial conditions.

We set the following boundary conditions and initial conditions in this paper:

$$\begin{aligned}
&\sigma^{\mathrm{por}} \cdot \mathbf{n} = -p\mathbf{n}, \quad \forall \mathbf{x} \in \mathcal{C}, \quad \mathbf{w} = \mathbf{0}, \quad \mathbf{u} \cdot \mathbf{n} = 0, \qquad \forall \mathbf{x} \in \partial\Omega, \\
&(\alpha\nabla \cdot \mathbf{w} + \phi)\rho_f(0) = (\alpha\nabla \cdot \mathbf{w}^0 + \phi)\rho_f^0.
\end{aligned}$$

For ease of presentation, we assume the gravity acceleration $\mathbf{g} = 0$ in the rest of the paper.

3 Variational Formulation

3.1 Space

Next, we present the variational formulation. We shall use the standard notations and definitions for Sobolev spaces[1] throughout the paper. Let Γ be a part of $\partial\Omega$ with positive measure.

$$W := \{v \in L^2(\Omega); v^\star \in H^1(\Omega \setminus \mathcal{C}), \star = \Omega \setminus \mathcal{C}\},$$

normed by the graph norm: $\|v\|_W^2 = \|v^\star\|_{H^1(\Omega\setminus\mathcal{C})}^2$.

The space for the displacement is

$$V = \{\mathbf{v} \in W^d; [\mathbf{v}]_{\Gamma\setminus\mathcal{C}} = 0, \mathbf{v}^\star_{|\partial\Omega} = 0, \star = \Omega \setminus \mathcal{C}\}, \tag{10}$$

with the norm of W^d: $\|\mathbf{v}\|_V^2 = \sum_{i=1}^d \|v_i\|_W^2$.

We denote the space $H^1(\mathcal{C})$:

$$H^1(\mathcal{C}) = \{z \in L^2(\mathcal{C}); \nabla_{\mathcal{C}} z \in L^2(\mathcal{C})^{d-1}\},$$

equipped with the norm: $\|z\|_{H^1(\mathcal{C})}^2 = \|z\|_{H^{\frac{1}{2}}(\mathcal{C})}^2 + \|\nabla_{\mathcal{C}} z\|_{L^2(\mathcal{C})}^2$.

We can specify the pressure space Q :

$$Q := \{q \in H^1(\Omega); q_c \in H^1(\mathcal{C})\}, \quad \text{where } q_c = q_{|\mathcal{C}}, \tag{11}$$

equipped with the graph norm: $\|q\|_Q^2 = \|q\|_{H^1(\Omega)}^2 + \|q_c\|_{H^1(\mathcal{C})}^2$.

The space of traces of functions of $H^1(\Omega)$ on Γ (or on any Lipschitz curve in Ω) is $H^{\frac{1}{2}}(\Gamma)$, which is a proper subspace of $L^2(\Gamma)$. Its dual space is denoted by $H^{-\frac{1}{2}}(\Gamma)$. The space for the width of the fracture is

$$\Theta_{\mathcal{C}} := H^{\frac{1}{2}}(\mathcal{C}). \tag{12}$$

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3.2 Weak Formulation

Let $\mathbf{w}$, ρ_f, and z be a sufficiently smooth solution of the coupled model, and let $\mathbf{v} \in V$ and $\theta \in Q$ be test displacement and test density functions, respectively. First, we take the scalar product of (1) in $\Omega \setminus \mathcal{C}$ with $\mathbf{v}$, apply Greens formula separately in $\Omega \setminus \mathcal{C}$, use the symmetry of σ and the continuity of $\sigma^{\mathrm{por}} \cdot \mathbf{n}^+$ on $\mathcal{C}$:

$$(\lambda(\nabla \cdot \mathbf{w}), \nabla \cdot \mathbf{v})_{\Omega\setminus\mathcal{C}} + (2G\epsilon(\mathbf{w}), \epsilon(\mathbf{v}))_{\Omega\setminus\mathcal{C}} - (\alpha p, \nabla \cdot \mathbf{v})_{\Omega\setminus\mathcal{C}} - (\sigma^{\mathrm{por}} \cdot \mathbf{n}^+, [\mathbf{v}])_{\mathcal{C}} = 0.$$

Noting that

$$[\mathbf{v}]_{\mathcal{C}} = ([\mathbf{v}]_{\mathcal{C}} \cdot \mathbf{n}^+)\mathbf{n}^+ + \sum_{j=1}^{d-1}([\mathbf{v}]_{\mathcal{C}} \cdot \tau_j^+)\tau_j^+,$$

and using (5), we write

$$\begin{aligned}
-(\sigma^{\mathrm{por}} \cdot \mathbf{n}^+, [\mathbf{v}])_{\mathcal{C}} &= -(\sigma^{\mathrm{por}}(\mathbf{w}, p)\mathbf{n}^+ \cdot \mathbf{n}^+, [\mathbf{v}] \cdot \mathbf{n}^+)_{\mathcal{C}} \\
&\quad - \sum_{j=1}^{d-1}(\sigma^{\mathrm{por}}(\mathbf{w}, p)\mathbf{n}^+ \cdot \tau_j^+, [\mathbf{v}] \cdot \tau_j^+)_{\mathcal{C}} \\
&= (p, [\mathbf{v}] \cdot \mathbf{n}^+)_{\mathcal{C}}.
\end{aligned}$$

This gives the variational equation of the displacement: Find $\mathbf{w} \in L^\infty(0, T; V)$, such that

$$(\lambda(\nabla \cdot \mathbf{w}), \nabla \cdot \mathbf{v})_{\Omega\setminus\mathcal{C}} + (2G\epsilon(\mathbf{w}), \epsilon(\mathbf{v}))_{\Omega\setminus\mathcal{C}} - (\alpha p, \nabla \cdot \mathbf{v})_{\Omega\setminus\mathcal{C}} + (p, [\mathbf{v}] \cdot \mathbf{n}^+)_{\mathcal{C}} = 0. \tag{13}$$

Next, we take the scalar product of (6) in $\Omega \setminus \mathcal{C}$ with $\theta \in Q$, apply Greens formula separately in $\Omega \setminus \mathcal{C}$, and use the continuity of θ through $\mathcal{C}$: Find $\rho_f \in L^\infty(0, T; L^2(\Omega)) \cap L^\infty(0, T; Q)$, such that

$$\begin{aligned}
&(\rho_f \frac{\partial(\alpha \nabla \cdot \mathbf{w})}{\partial t}, \theta)_{\Omega\setminus\mathcal{C}} + ((\alpha \nabla \cdot \mathbf{w} + \phi)\rho_f c_f \frac{\partial p}{\partial t}, \theta)_{\Omega\setminus\mathcal{C}} \\
&+ (\rho_f \cdot \frac{\mathbf{K}}{\eta} \nabla p, \nabla\theta)_{\Omega\setminus\mathcal{C}} - (\rho_f [\frac{\mathbf{K}}{\eta} \nabla p] \cdot \mathbf{n}^+, \theta)_{\mathcal{C}} = (q, \theta)_{\Omega\setminus\mathcal{C}}.
\end{aligned} \tag{14}$$

Finally, the third variational equation is obtained by taking the scalar product of (8) with $\theta_c \in \Theta_{\mathcal{C}}$, applying Greens formula in $\mathcal{C}$, and taking the time derivative out of the integral, and we obtain:

$$\begin{aligned}
(-\rho_f \frac{\partial([\mathbf{w}] \cdot \mathbf{n}^+)}{\partial t}, \theta_c)_{\mathcal{C}} - ([\mathbf{w}] \cdot \mathbf{n}^+ \rho_f c_f \frac{\partial p}{\partial t}, \theta_c)_{\mathcal{C}} + (\rho_f \cdot \frac{\mathbf{K}_{\mathrm{frac}}}{\eta} \nabla p, \nabla\theta_c)_{\mathcal{C}} \\
+(\rho_f \mathbf{u}^{\mathrm{frac}} \cdot \mathbf{n}, \theta_c)_{\mathcal{C}} + (\rho_f z, \theta_c)_{\mathcal{C}} = (q^{\mathrm{frac}}, \theta_c)_{\mathcal{C}}.
\end{aligned} \tag{15}$$

Remark 1. when $w_{\mathrm{frac}} = 0$ on the boundary of $\mathcal{C}$, the boundary term vanishes. In other words, our paper discusses the scenario under the assumption shown in Figure 2(*a*), excluding the situation in Figure 2(*b*).

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(a) Type 1 (b) Type 2

Fig. 2: The fracture cross-sectional diagram.

We again derive the third variational formulation as:
Find $z \in L^2(0,T;H^{-\frac{1}{2}}(\mathcal{C}))$, such that

$$(-\rho_f \frac{\partial([\mathbf{w}] \cdot \mathbf{n}^+)}{\partial t}, \theta_c)_{\mathcal{C}} - ([\mathbf{w}] \cdot \mathbf{n}^+ \rho_f c_f \frac{\partial p}{\partial t}, \theta_c)_{\mathcal{C}} + (\rho_f \cdot \frac{\mathbf{K}_{\text{frac}}}{\eta} \nabla p, \nabla \theta_c)_{\mathcal{C}} + (\rho_f z, \theta_c)_{\mathcal{C}} = (q^{\text{frac}}, \theta_c)_{\mathcal{C}}. \tag{16}$$

4 Energy

In this section, we first introduce the total free energy of the model and then derive the energy dissipation law for the model in the continuous case. According to the second law of thermodynamics, the total free energy in a closed system will dissipate over time[10].

4.1 Total Free Energy Definitions

We describe the Helmholtz free energy density determined by the Peng-Robinson equation of state [12]. The temperature is constant in the entire fluid-solid system. The Helmholtz free energy density $f(\rho_f)$ is expressed as a function of molar density ρ_f as follows:

$$f(\rho_f) = f_{\text{ide}}(\rho_f) + f_{\text{rep}}(\rho_f) + f_{\text{att}}(\rho_f), \tag{17}$$

$$f_{\text{ide}}(\rho_f) = \rho_f RT \ln(\rho_f), \tag{18}$$

$$f_{\text{rep}}(\rho_f) = -\rho_f RT \ln(1 - \beta_2 \rho_f), \tag{19}$$

$$f_{\text{att}}(\rho_f) = \frac{\beta_1(T)\rho_f}{2\sqrt{2}\beta_2} \ln\left(\frac{1 + (1-\sqrt{2})\beta_2\rho_f}{1 + (1+\sqrt{2})\beta_2\rho_f}\right), \tag{20}$$

where ρ_f is the molar density, T is the temperature, and R refers to the universal gas constant. Here, f_{ide} represents the free energy density of homogeneous ideal gas, while f_{rep} and f_{att} stand for the free energy contributions from the intermolecular repulsion and attraction effects, respectively. Let $T_r = T/T_c$ be the reduced temperature, where T_c is the critical temperature. The parameters β_1 and β_2 can be determined by the critical properties and the acentric factor,

$$\beta_1 = 0.45724 \frac{R^2 T_c^2}{P_c} \left[1 + m(1 - \sqrt{T_r})\right]^2, \quad \beta_2 = 0.07780 \frac{RT_c}{P_c}, \tag{21}$$

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where P_c stands for the critical pressure and m is calculated from the acentric factor ω as follows:

$$m = 0.37464 + 1.54226\omega - 0.26992\omega^2, \qquad \omega \le 0.49, \tag{22}$$
$$m = 0.379642 + 1.485030\omega - 0.164423\omega^2 + 0.016666\omega^3, \qquad \omega > 0.49. \tag{23}$$

The chemical potential, denoted by μ, is defined as the derivative of $f(\rho_f)$ with respect to molar density:

$$\mu(\rho_f) = f'(\rho_f). \tag{24}$$

We describe the following relationship between pressure, Helmholtz free energy density, and chemical potential [4,10,12]:

$$p = \rho_f \mu(\rho_f) - f(\rho_f). \tag{25}$$

We can derive the relation between the pressure gradient and the chemical potential gradient as

$$\nabla p = \nabla(\rho_f \mu(\rho_f) - f(\rho_f)) = \rho_f \nabla \mu(\rho_f) + \mu(\rho_f)\nabla\rho_f - \mu(\rho_f)\nabla\rho_f = \rho_f \nabla\mu(\rho_f). \tag{26}$$

We define the total free energy E_{tot} within the system as

$$E_{\text{tot}}(t) = H(t) + E_f(t) + E_{\text{frac}}(t), \tag{27}$$

where

$$H(t) = \int_{\Omega\setminus\mathcal{C}} H(\mathbf{w})\mathrm{d}\mathbf{x} = \frac{1}{2}\int_{\Omega\setminus\mathcal{C}} \sigma(\mathbf{w}) : \varepsilon(\mathbf{w})\mathrm{d}\mathbf{x}, \tag{28}$$
$$E_f(t) = \int_{\Omega\setminus\mathcal{C}} \phi_{\text{eff}} f(\rho_f)\mathrm{d}\mathbf{x}, \quad E_{\text{frac}}(t) = \int_{\mathcal{C}} w_{\text{frac}} f(\rho_f)\mathrm{d}\mathbf{s}, \tag{29}$$

we assume that $\phi_{\text{eff}} = \alpha\nabla\cdot\mathbf{w} + \phi > 0$.

4.2 Energy dissipation law

We now derive the equation for the variation of total free energy with respect to time. For the total solid elastic energy $H(t)$, we deduce that

$$\begin{aligned}\frac{\partial H(t)}{\partial t} &= \frac{1}{2}\int_{\Omega\setminus\mathcal{C}} \frac{\partial(\sigma(\mathbf{w}) : \varepsilon(\mathbf{w}))}{\partial t}\mathrm{d}\mathbf{x} \\ &= \int_{\Omega\setminus\mathcal{C}} \sigma(\mathbf{w}) : \nabla\frac{\partial\mathbf{w}}{\partial t}\mathrm{d}\mathbf{x} = -\int_{\Omega\setminus\mathcal{C}} \nabla\cdot\sigma(\mathbf{w})\frac{\partial\mathbf{w}}{\partial t}\mathrm{d}\mathbf{x}.\end{aligned} \tag{30}$$

For the variational equation of the displacement (13), taking the test function $\mathbf{v} = \frac{\partial\mathbf{w}}{\partial t}$, we have,

$$0 = (\lambda(\nabla\cdot\mathbf{w})\mathbf{I}, \nabla\frac{\partial\mathbf{w}}{\partial t})_{\Omega\setminus\mathcal{C}} + (2G\epsilon(\mathbf{w}), \nabla\frac{\partial\mathbf{w}}{\partial t})_{\Omega\setminus\mathcal{C}}$$

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$$
\begin{aligned}
&-(\alpha p\mathbf{I},\nabla\frac{\partial \mathbf{w}}{\partial t})_{\Omega\backslash\mathcal{C}}+(p,[\nabla\frac{\partial \mathbf{w}}{\partial t}]\cdot\mathbf{n}^+)_{\mathcal{C}}\\
&=-\int_{\Omega\backslash\mathcal{C}}\nabla\cdot\sigma(\mathbf{w})\frac{\partial \mathbf{w}}{\partial t}\mathrm{d}\mathbf{x}-\int_{\Omega\backslash\mathcal{C}}p\frac{\partial(\alpha\nabla\cdot\mathbf{w})}{\partial t}\mathrm{d}\mathbf{x}+\int_{\mathcal{C}}p\frac{\partial[\mathbf{w}]\cdot\mathbf{n}^+}{\partial t}\mathrm{d}\mathbf{s}.
\end{aligned}
$$

Thus, we can get the following equation,

$$
\frac{\partial H(t)}{\partial t}=\underbrace{\int_{\Omega\backslash\mathcal{C}}p\frac{\partial(\alpha\nabla\cdot\mathbf{w})}{\partial t}\mathrm{d}\mathbf{x}}_{①}\underbrace{-\int_{\mathcal{C}}p\frac{\partial[\mathbf{w}]\cdot\mathbf{n}^+}{\partial t}\mathrm{d}\mathbf{s}}_{②}. \tag{31}
$$

Noting these equations that $\mathbf{u}=-\frac{\mathbf{K}}{\eta}\nabla p$, $\frac{\partial\phi}{\partial t}=0$, $p=\rho_f\mu(\rho_f)-f(\rho_f)$ and $\nabla p=\rho_f\nabla\mu(\rho_f)$, the variation of $E_f(t)$ with time is derived as

$$
\begin{aligned}
\frac{\partial E_f(t)}{\partial t}&=\int_{\Omega\backslash\mathcal{C}}\frac{\partial\phi_{\mathrm{eff}}f(\rho_f)}{\partial t}\mathrm{d}\mathbf{x}=\int_{\Omega\backslash\mathcal{C}}\frac{\partial(\alpha\nabla\cdot\mathbf{w}+\phi)f(\rho_f)}{\partial t}\mathrm{d}\mathbf{x} &&(32)\\
&=\int_{\Omega\backslash\mathcal{C}}f(\rho_f)\frac{\partial(\alpha\nabla\cdot\mathbf{w}+\phi)}{\partial t}\mathrm{d}\mathbf{x}+\int_{\Omega\backslash\mathcal{C}}(\alpha\nabla\cdot\mathbf{w}+\phi)\frac{\partial f(\rho_f)}{\partial t}\mathrm{d}\mathbf{x}\\
&=\int_{\Omega\backslash\mathcal{C}}f(\rho_f)\frac{\partial(\alpha\nabla\cdot\mathbf{w}+\phi)}{\partial t}\mathrm{d}\mathbf{x}-\int_{\Omega\backslash\mathcal{C}}\frac{\partial(\alpha\nabla\cdot\mathbf{w}+\phi)}{\partial t}\mu(\rho_f)\rho_f\mathrm{d}\mathbf{x}\\
&+\int_{\Omega\backslash\mathcal{C}}\frac{\partial(\alpha\nabla\cdot\mathbf{w}+\phi)\rho_f}{\partial t}\mu(\rho_f)\mathrm{d}\mathbf{x}\\
&=\int_{\Omega\backslash\mathcal{C}}(f(\rho_f)-\mu(\rho_f)\rho_f)\frac{\partial(\alpha\nabla\cdot\mathbf{w}+\phi)}{\partial t}\mathrm{d}\mathbf{x}\\
&+\int_{\Omega\backslash\mathcal{C}}\frac{\partial(\alpha\nabla\cdot\mathbf{w}+\phi)\rho_f}{\partial t}\mu(\rho_f)\mathrm{d}\mathbf{x}\\
&=-\int_{\Omega\backslash\mathcal{C}}p\frac{\partial(\alpha\nabla\cdot\mathbf{w}+\phi)}{\partial t}\mathrm{d}\mathbf{x}+\int_{\Omega\backslash\mathcal{C}}(q-\nabla\cdot(\rho_f\mathbf{u}))\mu(\rho_f)\mathrm{d}\mathbf{x}\\
&=-\int_{\Omega\backslash\mathcal{C}}p\frac{\partial(\alpha\nabla\cdot\mathbf{w})}{\partial t}\mathrm{d}\mathbf{x}+\int_{\Omega\backslash\mathcal{C}}q\mu(\rho_f)\mathrm{d}\mathbf{x}-\int_{\Omega\backslash\mathcal{C}}\frac{\mathbf{K}}{\eta}\rho_f\nabla\mu(\rho_f)\nabla p\mathrm{d}\mathbf{x}\\
&+\int_{\mathcal{C}}\mu(\rho_f)\rho_f\left[\frac{\mathbf{K}}{\eta}\nabla p\right]\cdot\mathbf{n}^+\mathrm{d}\mathbf{s}\\
&=\underbrace{-\int_{\Omega\backslash\mathcal{C}}p\frac{\partial(\alpha\nabla\cdot\mathbf{w})}{\partial t}\mathrm{d}\mathbf{x}}_{①}\underbrace{+\int_{\mathcal{C}}\mu(\rho_f)\rho_f\left[\frac{\mathbf{K}}{\eta}\nabla p\right]\cdot\mathbf{n}^+\mathrm{d}\mathbf{s}}_{③}\\
&+\int_{\Omega\backslash\mathcal{C}}q\mu(\rho_f)\mathrm{d}\mathbf{x}-\int_{\Omega\backslash\mathcal{C}}\frac{\mathbf{K}}{\eta}|\rho_f(\nabla\mu(\rho_f))|^2\mathrm{d}\mathbf{x}.
\end{aligned}
$$

For flow equation in the fracture, we note that $w_{\mathrm{frac}}=-[\mathbf{w}]_{\mathcal{C}}\cdot\mathbf{n}^+,\mathbf{u}^{\mathrm{frac}}=-\frac{\mathbf{K}_{\mathrm{frac}}}{\eta}\nabla p$, $z=-[\mathbf{u}]_{\mathcal{C}}\cdot\mathbf{n}^+$, and can derive,

$$
\frac{\partial E_{\mathrm{frac}}(t)}{\partial t}=\int_{\mathcal{C}}\frac{\partial w_{\mathrm{frac}}f(\rho_f)}{\partial t}\mathrm{d}\mathbf{s}=\int_{\mathcal{C}}w_{\mathrm{frac}}\frac{\partial f(\rho_f)}{\partial t}\mathrm{d}\mathbf{s}+\int_{\mathcal{C}}f(\rho_f)\frac{\partial w_{\mathrm{frac}}}{\partial t}\mathrm{d}\mathbf{s} \tag{33}
$$

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$$
\begin{aligned}
&= \int_{\mathcal{C}} w_{\text{frac}} \mu(\rho_f) \frac{\partial \rho_f}{\partial t} \mathrm{d}\mathbf{s} + \int_{\mathcal{C}} (\rho_f \mu(\rho_f) - p) \frac{\partial w_{\text{frac}}}{\partial t} \mathrm{d}\mathbf{s} \\
&= \int_{\mathcal{C}} (w_{\text{frac}} \mu(\rho_f) \frac{\partial \rho_f}{\partial t} + \rho_f \mu(\rho_f) \frac{\partial w_{\text{frac}}}{\partial t}) \mathrm{d}\mathbf{s} - \int_{\mathcal{C}} p \frac{\partial w_{\text{frac}}}{\partial t} \mathrm{d}\mathbf{s} \\
&= \int_{\mathcal{C}} \mu(\rho_f) \frac{\partial (w_{\text{frac}} \rho_f)}{\partial t} \mathrm{d}\mathbf{s} - \int_{\mathcal{C}} p \frac{\partial w_{\text{frac}}}{\partial t} \mathrm{d}\mathbf{s} \\
&= \int_{\mathcal{C}} \mu(\rho_f)(q^{\text{frac}} - \rho_f z - \nabla_{\mathcal{C}} \cdot (\rho_f \mathbf{u}^{\text{frac}})) \mathrm{d}\mathbf{s} - \int_{\mathcal{C}} p \frac{\partial w_{\text{frac}}}{\partial t} \mathrm{d}\mathbf{s} \\
&= \underbrace{\int_{\mathcal{C}} p \frac{\partial [\mathbf{w}] \cdot \mathbf{n}^+}{\partial t} \mathrm{d}\mathbf{s}}_{②} \underbrace{- \int_{\mathcal{C}} \mu(\rho_f) \rho_f \left[\frac{\mathbf{K}}{\eta} \nabla p \right] \cdot \mathbf{n}^+ \mathrm{d}\mathbf{s}}_{③} \\
&+ \int_{\mathcal{C}} \mu(\rho_f) q^{\text{frac}} \mathrm{d}\mathbf{s} - \int_{\mathcal{C}} \frac{\mathbf{K}_{\text{frac}}}{\eta} |\rho_f (\nabla \mu(\rho_f))|^2 \mathrm{d}\mathbf{s}.
\end{aligned}
$$

Since the total free energy is always decreasing over time, in general there should exist a phenomenological coefficient $\eta_s \geq 0$ such that $-\nabla \cdot \sigma(\mathbf{w}) = \nabla \cdot \eta_s \nabla \mathbf{w}_s$, where $\mathbf{w}_s = \frac{\partial \mathbf{w}}{\partial t}$, which ensures that the right-hand side of (30) is always less than zero as follows

$$
-\int_{\Omega \setminus \mathcal{C}} \nabla \cdot \sigma(\mathbf{w}) \frac{\partial \mathbf{w}}{\partial t} \mathrm{d}\mathbf{x} = \int_{\Omega \setminus \mathcal{C}} \nabla \cdot \eta_s \nabla \mathbf{w}_s \cdot \mathbf{w}_s \mathrm{d}\mathbf{x} = -\int_{\Omega \setminus \mathcal{C}} \eta_s \nabla \mathbf{w}_s : \nabla \mathbf{w}_s \mathrm{d}\mathbf{x} \leq 0.
$$

Let's add the above equations (31) − (33) together, we can deduce that the model obeys the energy dissipation law within a closed system,

$$
\begin{aligned}
\frac{\partial E_{\text{tot}}}{\partial t} &= \frac{\partial H(t)}{\partial t} + \frac{\partial E_f(t)}{\partial t} + \frac{\partial E_{\text{frac}}(t)}{\partial t} \\
&= \int_{\Omega \setminus \mathcal{C}} q \mu(\rho_f) \mathrm{d}\mathbf{x} - \int_{\Omega \setminus \mathcal{C}} \frac{\mathbf{K}}{\eta} |\rho_f (\nabla \mu(\rho_f))|^2 \mathrm{d}\mathbf{x} \\
&+ \int_{\mathcal{C}} \mu(\rho_f) q^{\text{frac}} \mathrm{d}\mathbf{s} - \int_{\mathcal{C}} \frac{\mathbf{K}_{\text{frac}}}{\eta} |\rho_f (\nabla \mu(\rho_f))|^2 \mathrm{d}\mathbf{s},
\end{aligned} \tag{34}
$$

which indicates that the total energy is always dissipated with time.

Theorem 1. *For the closed system with the boundary conditions* $\mathbf{w} = \mathbf{0}$ *and* $\mathbf{u} \cdot \mathbf{n} = 0$ *on the boundary* $\partial\Omega$*, where* $\mathbf{n}$ *denotes the normal unit outward vector to* $\partial\Omega$*, the gravity* $\mathbf{g} = \mathbf{0}$*,* $q = 0$ *and* $q^{\text{frac}} = 0$*. The coupled model* (P) *satisfies the following discrete energy dissipation law as*

$$
\frac{\partial E_{\text{tot}}}{\partial t} + \int_{\Omega \setminus \mathcal{C}} \frac{\mathbf{K}}{\eta} |\rho_f (\nabla \mu(\rho_f))|^2 \mathrm{d}\mathbf{x} + \int_{\mathcal{C}} \frac{\mathbf{K}_{\text{frac}}}{\eta} |\rho_f (\nabla \mu(\rho_f))|^2 \mathrm{d}\mathbf{s} \leq 0. \tag{35}
$$

5 Time-Discrete Scheme

The energy dissipation law is a fundamental principle followed by nature, which is also inherent in our proposed model. Therefore, an effective numerical

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method should maintain this law. In this section, we develop an energy-stable time-discrete scheme. We divide the time interval $[0, T]$ into N time steps as $0 = t_0 < t_1 < \cdots < t_N = T$ and denote the time step size by $\Delta t = t^{n+1} - t^n$. For any variable v, the superscript n in v^n indicates the approximation of v at the time t^n.

5.1 Implicit scheme

We propose the following implicit scheme

$$-\nabla \cdot (\lambda(\nabla \cdot \mathbf{w}^{n+1})\mathbf{I} + 2G\epsilon(\mathbf{w}^{n+1}) - \alpha p^{n+1}\mathbf{I}) = 0, \quad \forall \mathbf{x} \in \Omega \setminus \mathcal{C}, \tag{36}$$

$$(\sigma^{\mathrm{por}})^{n+1} \cdot \mathbf{n} = -p^{n+1}\mathbf{n}, \quad \forall \mathbf{x} \in \mathcal{C}, \tag{37}$$

$$\frac{(\phi + \alpha\nabla \cdot \mathbf{w}^{n+1})\rho_f^{n+1} - (\phi + \alpha\nabla \cdot \mathbf{w}^n)\rho_f^n}{\Delta t} + \nabla \cdot (\rho_f^{n+1}\mathbf{u}^{n+1}) = q^{n+1}, \ \forall \mathbf{x} \in \Omega \setminus \mathcal{C}, \tag{38}$$

$$\mathbf{u}^{n+1} = -\frac{\mathbf{K}}{\eta}\nabla p^{n+1}, \quad \forall \mathbf{x} \in \Omega \setminus \mathcal{C}, \tag{39}$$

$$\frac{w_{\mathrm{frac}}^{n+1}\rho_f^{n+1} - w_{\mathrm{frac}}^n\rho_f^n}{\Delta t} + \nabla_{\mathcal{C}} \cdot (\rho_f^{n+1}(\mathbf{u}^{\mathrm{frac}})^{n+1}) = (q^{\mathrm{frac}})^{n+1} + \rho_f^{n+1}\left[\mathbf{u}^{n+1}\right]_{\mathcal{C}} \cdot \mathbf{n}^+, \tag{40}$$

$$w_{\mathrm{frac}}^{n+1} = -\left[\mathbf{w}^{n+1}\right]_{\mathcal{C}} \cdot \mathbf{n}^+, \ \ (\mathbf{u}^{\mathrm{frac}})^{n+1} = -\frac{\mathbf{K}_{\mathrm{frac}}}{\eta}\nabla p^{n+1}, \quad \forall \mathbf{x} \in \mathcal{C}, \tag{41}$$

$$\frac{\mathrm{d}\rho_f}{\mathrm{d}p} = \frac{\rho_f^{n+1} - \rho_f^n}{p^{n+1} - p^n}, \ \frac{\rho_f^{n+1} - \rho_f^n}{p^{n+1} - p^n} = \rho_f^{n+1}c_f^{n+1}, \quad \forall \mathbf{x} \in \Omega. \tag{42}$$

In (36), we approximate the pressure of p^{n+1} using (42).

Remark 2. If the fluid compressibility c_f is assumed to be small (e.g., of the order of 10^{-8} or 10^{-9}), and $\rho_f c_f$ is also small. When we use (42) to calculate the pressure, $\frac{1}{\rho_f^{n+1}c_f^{n+1}}$ becomes a very large number, which causes a significant error in our pressure calculation. Therefore, this paper emphasizes that the applicability of the proposed model is based on the assumption that the fluid has a large compressibility coefficient.

5.2 Discrete energy dissipation law of the implicit scheme

We now show that the implicit scheme follows a discrete energy dissipation law. We first derive the discrete chemical potentials that preserve the energy stability at the time-discrete level. For $\rho_f^{n+1} > 0$ and $\rho_f^n > 0$, $\mu^{n+1} = \mu(\rho_f^{n+1})$, using the convexity of $f(\rho_f)$, we can deduce that

$$f(\rho_f^{n+1}) - f(\rho_f^{n+1}) \leq \mu^{n+1}(\rho_f^{n+1} - \rho_f^n). \tag{43}$$

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For the derivation of inequality (43), the detailed procedure can be found in the literatures[9,12]. We define the discrete total energy as

$$E_{\text{tot}}^{n+1} = H^{n+1} + E_f^{n+1} + (E_f^{\text{frac}})^{n+1}. \tag{44}$$

Using (28), we deduce that

$$\begin{aligned}
&H^{n+1} - H^n = \frac{1}{2}\int_{\Omega\setminus\mathcal{C}} \left(\sigma^{n+1} : \varepsilon(\mathbf{w}^{n+1}) - \sigma^n : \varepsilon(\mathbf{w}^n)\right) \mathrm{d}\mathbf{x} \\
&= \frac{1}{2}\int_{\Omega\setminus\mathcal{C}} \lambda\left(|\nabla\cdot\mathbf{w}^{n+1}|^2 - |\nabla\cdot\mathbf{w}^n|^2\right) \mathrm{d}\mathbf{x} \\
&+ \int_{\Omega\setminus\mathcal{C}} G\left(\varepsilon(\mathbf{w}^{n+1}) : \varepsilon(\mathbf{w}^{n+1}) - \varepsilon(\mathbf{w}^n) : \varepsilon(\mathbf{w}^n)\right) \mathrm{d}\mathbf{x} \\
&= \int_{\Omega\setminus\mathcal{C}} \lambda\nabla\cdot\mathbf{w}^{n+1}\nabla\cdot(\mathbf{w}^{n+1} - \mathbf{w}^n)\mathrm{d}\mathbf{x} - \frac{1}{2}\int_{\Omega\setminus\mathcal{C}} \lambda|\nabla\cdot(\mathbf{w}^{n+1} - \mathbf{w}^n)|^2\mathrm{d}\mathbf{x} \\
&+ \int_{\Omega\setminus\mathcal{C}} 2G\varepsilon(\mathbf{w}^{n+1}) : \left(\varepsilon(\mathbf{w}^{n+1}) - \varepsilon(\mathbf{w}^n)\right) \mathrm{d}\mathbf{x} - \int_{\Omega\setminus\mathcal{C}} G|\varepsilon(\mathbf{w}^{n+1}) - \varepsilon(\mathbf{w}^n)|^2\mathrm{d}\mathbf{x} \\
&\leq \int_{\Omega\setminus\mathcal{C}} \left(\lambda\nabla\cdot\mathbf{w}^{n+1}\cdot\boldsymbol{I} + 2G\boldsymbol{\varepsilon}(\mathbf{w}^{n+1})\right) : \nabla(\mathbf{w}^{n+1} - \mathbf{w}^n)\mathrm{d}\mathbf{x} \\
&= -\int_{\Omega\setminus\mathcal{C}} \nabla\cdot\sigma^{n+1}\cdot\left(\mathbf{w}^{n+1} - \mathbf{w}^n\right) \mathrm{d}\mathbf{x}.
\end{aligned}$$

For the variational expression of the displacement (13) in $t = t^{n+1}$, by choosing the test function $\mathbf{v} = \frac{\mathbf{w}^{n+1} - \mathbf{w}^n}{\Delta t}$, we can derive the following result,

$$\begin{aligned}
&0 = (-\nabla\cdot(\lambda(\nabla\cdot\mathbf{w}^{n+1})\mathbf{I} + 2G\epsilon(\mathbf{w}^{n+1}) - \alpha p^{n+1}\mathbf{I}), \frac{\mathbf{w}^{n+1} - \mathbf{w}^n}{\Delta t})_{\Omega\setminus\mathcal{C}} \qquad (45) \\
&= (\sigma^{n+1}, \nabla\frac{\mathbf{w}^{n+1} - \mathbf{w}^n}{\Delta t})_{\Omega\setminus\mathcal{C}} - (\alpha p^{n+1}\mathbf{I}, \nabla\frac{\mathbf{w}^{n+1} - \mathbf{w}^n}{\Delta t})_{\Omega\setminus\mathcal{C}} \\
&- ((\sigma^{\text{por}})^{n+1}\cdot\mathbf{n}^+, \frac{\mathbf{w}^{n+1} - \mathbf{w}^n}{\Delta t})_{\mathcal{C}} \\
&= (-\nabla\cdot\sigma^{n+1}, \frac{\mathbf{w}^{n+1} - \mathbf{w}^n}{\Delta t})_{\Omega\setminus\mathcal{C}} - (p^{n+1}, \frac{\alpha\nabla\cdot\mathbf{w}^{n+1} - \alpha\nabla\cdot\mathbf{w}^n}{\Delta t})_{\Omega\setminus\mathcal{C}} \\
&+ (p^{n+1}, \frac{\left[\mathbf{w}^{n+1}\right]\cdot\mathbf{n}^+ - \left[\mathbf{w}^n\right]\cdot\mathbf{n}^+}{\Delta t})_{\mathcal{C}}.
\end{aligned}$$

Thus, we can obtain the following inequality,

$$\begin{aligned}
\frac{H^{n+1} - H^n}{\Delta t} \leq &\underbrace{\int_{\Omega\setminus\mathcal{C}} p^{n+1}\frac{\alpha\nabla\cdot\mathbf{w}^{n+1} - \alpha\nabla\cdot\mathbf{w}^n}{\Delta t}\mathrm{d}\mathbf{x}}_{①} \\
&\underbrace{-\int_{\mathcal{C}} (p^{n+1}\frac{\left[\mathbf{w}^{n+1}\right]\cdot\mathbf{n}^+ - \left[\mathbf{w}^n\right]\cdot\mathbf{n}^+}{\Delta t}\mathrm{d}\mathbf{s}.}_{②}
\end{aligned}$$

We use the properties of the energy function $f(\rho_f)$, for the equations of fluid flow in $\Omega \setminus \mathcal{C}$, can derive,

$$
\begin{aligned}
\frac{E_f^{n+1}-E_f^{n}}{\Delta t} &= \int_{\Omega\setminus\mathcal{C}} \frac{(\alpha\nabla\cdot\mathbf{w}^{n+1}+\phi)f(\rho_f^{n+1})-(\alpha\nabla\cdot\mathbf{w}^{n}+\phi)f(\rho_f^{n})}{\Delta t}\mathrm{d}\mathbf{x} \qquad (46)\\
&= \int_{\Omega\setminus\mathcal{C}} f(\rho_f^{n+1})\frac{\alpha\nabla\cdot\mathbf{w}^{n+1}-\alpha\nabla\cdot\mathbf{w}^{n}}{\Delta t}\mathrm{d}\mathbf{x}\\
&+ \int_{\Omega\setminus\mathcal{C}} (\alpha\nabla\cdot\mathbf{w}^{n}+\phi)\frac{f(\rho_f^{n+1})-f(\rho_f^{n})}{\Delta t}\mathrm{d}\mathbf{x}\\
&\le \int_{\Omega\setminus\mathcal{C}} (\rho_f^{n+1}\mu^{n+1}-p^{n+1})\frac{\alpha\nabla\cdot\mathbf{w}^{n+1}-\alpha\nabla\cdot\mathbf{w}^{n}}{\Delta t}\mathrm{d}\mathbf{x}\\
&+ \int_{\Omega\setminus\mathcal{C}} (\alpha\nabla\cdot\mathbf{w}^{n}+\phi)\mu^{n+1}\frac{\rho_f^{n+1}-\rho_f^{n}}{\Delta t}\mathrm{d}\mathbf{x}\\
&\le \int_{\Omega\setminus\mathcal{C}} \mu^{n+1}\frac{(\phi+\alpha\nabla\cdot\mathbf{w}^{n+1})\rho_f^{n+1}-(\phi+\alpha\nabla\cdot\mathbf{w}^{n})\rho_f^{n}}{\Delta t}\mathrm{d}\mathbf{x}\\
&- \int_{\Omega\setminus\mathcal{C}} p^{n+1}\frac{\alpha\nabla\cdot\mathbf{w}^{n+1}-\alpha\nabla\cdot\mathbf{w}^{n}}{\Delta t}\mathrm{d}\mathbf{x}\\
&\le \int_{\Omega\setminus\mathcal{C}} \mu^{n+1}(q^{n+1}-\nabla\cdot(\rho_f^{n+1}\mathbf{u}^{n+1}))\mathrm{d}\mathbf{x} - \int_{\Omega\setminus\mathcal{C}} p^{n+1}\frac{\alpha\nabla\cdot\mathbf{w}^{n+1}-\alpha\nabla\cdot\mathbf{w}^{n}}{\Delta t}\mathrm{d}\mathbf{x}\\
&\le \int_{\Omega\setminus\mathcal{C}} \mu^{n+1}q^{n+1}\mathrm{d}\mathbf{x} + \underbrace{\int_{\mathcal{C}} \mu^{n+1}\rho_f^{n+1}\left[\frac{\mathbf{K}}{\eta}\nabla p^{n+1}\right]\cdot\mathbf{n}^{+}\mathrm{d}\mathbf{s}}_{③}\\
&- \int_{\Omega\setminus\mathcal{C}} \frac{\mathbf{K}}{\eta}|\rho_f^{n+1}\nabla\mu^{n+1}|^2\mathrm{d}\mathbf{x} \underbrace{- \int_{\Omega\setminus\mathcal{C}} p^{n+1}\frac{\alpha\nabla\cdot\mathbf{w}^{n+1}-\alpha\nabla\cdot\mathbf{w}^{n}}{\Delta t}\mathrm{d}\mathbf{x}}_{①}.
\end{aligned}
$$

Using the equation for conservation of mass in a fracture $\mathcal{C}$, we derive the following energy inequality,

$$
\begin{aligned}
\frac{(E_f^{\mathrm{frac}})^{n+1}-(E_f^{\mathrm{frac}})^{n}}{\Delta t} &= \int_{\mathcal{C}} \frac{w_{\mathrm{frac}}^{n+1}f(\rho_f^{n+1})-w_{\mathrm{frac}}^{n}f(\rho_f^{n})}{\Delta t}\mathrm{d}\mathbf{s} \qquad (47)\\
&= \int_{\mathcal{C}} f(\rho_f^{n+1})\frac{w_{\mathrm{frac}}^{n+1}-w_{\mathrm{frac}}^{n}}{\Delta t}\mathrm{d}\mathbf{s} + \int_{\mathcal{C}} w_{\mathrm{frac}}^{n}\frac{f(\rho_f^{n+1})-f(\rho_f^{n})}{\Delta t}\mathrm{d}\mathbf{s}\\
&\le \int_{\mathcal{C}} (\rho_f^{n+1}\mu^{n+1}-p^{n+1})\frac{w_{\mathrm{frac}}^{n+1}-w_{\mathrm{frac}}^{n}}{\Delta t}\mathrm{d}\mathbf{s} + \int_{\mathcal{C}} w_{\mathrm{frac}}^{n}\mu^{n+1}\frac{\rho_f^{n+1}-\rho_f^{n}}{\Delta t}\mathrm{d}\mathbf{s}\\
&\le \int_{\mathcal{C}} \mu^{n+1}\frac{w_{\mathrm{frac}}^{n+1}\rho_f^{n+1}-w_{\mathrm{frac}}^{n}\rho_f^{n}}{\Delta t}\mathrm{d}\mathbf{s} + \int_{\mathcal{C}} p^{n+1}\frac{[\mathbf{w}^{n+1}]_{\mathcal{C}}\cdot\mathbf{n}^{+}-[\mathbf{w}^{n}]_{\mathcal{C}}\cdot\mathbf{n}^{+}}{\Delta t}\mathrm{d}\mathbf{s}\\
&\le \int_{\mathcal{C}} \mu^{n+1}((q^{\mathrm{frac}})^{n+1}+\rho_f^{n+1}\left[\mathbf{u}^{n+1}\right]_{\mathcal{C}}\cdot\mathbf{n}^{+}-\nabla_{\mathcal{C}}\cdot(\rho_f^{n+1}(\mathbf{u}^{\mathrm{frac}})^{n+1}))\mathrm{d}\mathbf{s}
\end{aligned}
$$

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$$
\begin{aligned}
&+\int_{\mathcal{C}} p^{n+1}\frac{\left[\mathbf{w}^{n+1}\right]_{\mathcal{C}}\cdot\mathbf{n}^{+}-\left[\mathbf{w}^{n}\right]_{\mathcal{C}}\cdot\mathbf{n}^{+}}{\Delta t}\mathrm{d}\mathbf{s}\\
&\leq \int_{\mathcal{C}} \mu^{n+1}(q^{\mathrm{frac}})^{n+1}\mathrm{d}\mathbf{s}\underbrace{-\int_{\mathcal{C}}\mu^{n+1}\rho_f^{n+1}\left[\frac{\mathbf{K}}{\eta}\nabla p^{n+1}\right]\cdot\mathbf{n}^{+}\mathrm{d}\mathbf{s}}_{③}\\
&-\int_{\mathcal{C}}\frac{\mathbf{K}_{\mathrm{frac}}}{\eta}|\rho_f^{n+1}\nabla\mu^{n+1}|^2\mathrm{d}\mathbf{s}\underbrace{+\int_{\mathcal{C}} p^{n+1}\frac{\left[\mathbf{w}^{n+1}\right]_{\mathcal{C}}\cdot\mathbf{n}^{+}-\left[\mathbf{w}^{n}\right]_{\mathcal{C}}\cdot\mathbf{n}^{+}}{\Delta t}\mathrm{d}\mathbf{s}}_{②}.
\end{aligned}
$$

The implicit scheme satisfies the following discrete energy dissipation law:

$$
\begin{aligned}
\frac{E_{\mathrm{tot}}^{n+1}-E_{\mathrm{tot}}^{n}}{\Delta t} &= \frac{H^{n+1}-H^{n}}{\Delta t}+\frac{E_f^{n+1}-E_f^{n}}{\Delta t}+\frac{(E_f^{\mathrm{frac}})^{n+1}-(E_f^{\mathrm{frac}})^{n}}{\Delta t}\\
&\leq \int_{\Omega\setminus\mathcal{C}}\mu^{n+1}q^{n+1}\mathrm{d}\mathbf{x}-\int_{\Omega\setminus\mathcal{C}}\frac{\mathbf{K}}{\eta}|\rho_f^{n+1}\nabla\mu^{n+1}|^2\mathrm{d}\mathbf{x}\\
&+\int_{\mathcal{C}}\mu^{n+1}(q^{\mathrm{frac}})^{n+1}\mathrm{d}\mathbf{s}-\int_{\mathcal{C}}\frac{\mathbf{K}_{\mathrm{frac}}}{\eta}|\rho_f^{n+1}\nabla\mu^{n+1}|^2\mathrm{d}\mathbf{s},
\end{aligned}
\tag{48}
$$

which shows that the total discrete energy decreases with each time step.

Theorem 2. *For the closed system with the boundary conditions $\mathbf{w}^{n+1}=\mathbf{0}$ and $\mathbf{u}^{n+1}\cdot\mathbf{n}=0$ on the boundary $\partial\Omega$, where $\mathbf{n}$ denotes the normal unit outward vector to $\partial\Omega$, the gravity $\mathbf{g}=\mathbf{0}$, $q^{n+1}=0$ and $(q^{\mathrm{frac}})^{n+1}=0$. The implicit scheme satisfies the following discrete energy dissipation law:*

$$
\frac{E_{\mathrm{tot}}^{n+1}-E_{\mathrm{tot}}^{n}}{\Delta t}+\int_{\Omega\setminus\mathcal{C}}\frac{\mathbf{K}}{\eta}|\rho_f^{n+1}\nabla\mu^{n+1}|^2\mathrm{d}\mathbf{x}+\int_{\mathcal{C}}\frac{\mathbf{K}_{\mathrm{frac}}}{\eta}|\rho_f^{n+1}\nabla\mu^{n+1}|^2\mathrm{d}\mathbf{s}\leq 0. \tag{49}
$$

6 Conclusions

In conclusion, this paper presents a comprehensive investigation of the fluid-solid coupling problem in fractured porous elastic media, explicitly accounting for the geometry of fractures as potentially non-planar interfaces. The derived model, which involves mixed-dimensional equations, successfully couples the flow on the $d-1$ dimensional fracture surfaces with the d dimensional porous matrix. By adopting a strongly compressible fluid flow model with density as the primary variable, this work contrasts with previous studies that used pressure as the primary variable for slightly compressible fluids. A thermodynamically consistent mathematical model is developed, and its weak formulation is provided. Additionally, energy stability is rigorously established for both the continuous and semi-discrete formulations in time. The proposed model and numerical framework offer a robust and stable platform for simulating strongly compressible fluid flows, ensuring both thermodynamic consistency and computational reliability.

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References

1. Robert Adams and John Fournier. Sobolev spaces. 140:xiv+305, 2003.
2. Tameem Almani and Kundan Kumar. Convergence of single rate and multirate undrained split iterative schemes for a fractured biot model. *Computational Geosciences*, 26(4):975–994, 2022.
3. Tameem Almani and Kundan Kumar. Convergence of multirate fixed stress split iterative schemes for a fractured biot model. *Computer Methods in Applied Mechanics and Engineering*, 431:117253, 2024.
4. Xiaolin Fan, Jisheng Kou, Zhonghua Qiao, and Shuyu Sun. A componentwise convex splitting scheme for diffuse interface models with van der waals and peng–robinson equations of state. *SIAM Journal on Scientific Computing*, 39(1):B1–B28, 2017.
5. Vivette Girault, Mary Wheeler, Benjamin Ganis, and Mark Mear. A lubrication fracture model in a poro-elastic medium. *Mathematical Models and Methods in Applied Sciences*, 25(4):587–645, 2015.
6. B. J. Hamrock, S. R. Schmid, and B. O. Jacobson. Fundamentals of fluid film lubrication. *Crc Press*, 2004.
7. Birendra Jha and Ruben Juanes. Coupled multiphase flow and poromechanics: A computational model of pore pressure effects on fault slip and earthquake triggering. *Water Resources Research*, 50(5):3776–3808, 2014.
8. Jisheng Kou, Huangxin Chen, Amgad Salama, and Shuyu Sun. An energy stable and positivity-preserving computational method for compressible and immiscible two-phase flow in porous media. *Journal of Computational Physics*, 519:113391, 2024.
9. Jisheng Kou, Amgad Salama, Huangxin Chen, and Shuyu Sun. Thermodynamically consistent numerical modeling of immiscible two-phase flow in poro-viscoelastic media. *International Journal for Numerical Methods in Engineering*, 125(14):e7479, jul 2024. First published: 08 April 2024.
10. Jisheng Kou and Shuyu Sun. Thermodynamically consistent modeling and simulation of multi-component two-phase flow with partial miscibility. *Computer Methods in Applied Mechanics and Engineering*, 331(APR.1):623–649, 2018.
11. Jisheng Kou, Xiuhua Wang, and ShiGui Du. Energy stable and mass conservative numerical method for gas flow in porous media with rock compressibility. *SIAM Journal on Scientific Computing*, 44(4):B938–B963, 2022.
12. Jisheng Kou, Xiuhua Wang, and ShiGui Du. Energy stable and mass conservative numerical method for gas flow in porous media with rock compressibility. *SIAM Journal on Scientific Computing*, 44(4):B938–B963, 2022.
13. M. J. Martinez, P. Newell, J. E. Bishop, and D. Z. Turner. Coupled multiphase flow and geomechanics model for analysis of joint reactivation during co_2 sequestration operations. *International Journal of Greenhouse Gas Control*, 17:148–160, 2013.
14. Paul A. Witherspoon, Joseph S. Y. Wang, Kazuhiro Iwai, and John E. Gale. Validity of cubic law for fluid flow in a deformable rock fracture. *Water Resources Research*, 16(6), 1980.
15. Chong Hooi Yew. Mechanics of hydraulic fracturing. 1997.

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