Asymptotics in Curve Estimation by Modified Cubic Spline and Exponential Parameterization

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Abstract. The problem of fitting reduced data Q_m is discussed here. Reduced data form the ordered sequence of interpolation points $q_i = \gamma(t_i)$ in arbitrary Euclidean space. Here the corresponding unknown knots $\mathcal{T}$ are replaced with $\hat{\mathcal{T}}$ compensated by the so-called exponential parameterization determined by reduced data Q_m and a single parameter $\lambda \in [0,1]$. In sequel, a modified complete spline is used to interpolate Q_m with the aid of exponential parameterization. The main theoretical contribution of this work is to prove a linear convergence order in γ estimation by fitting Q_m (getting denser) with modified complete spline based on exponential parameterization for $\lambda \in [0,1)$. The latter holds for sufficiently smooth, regular curves sampled more-or-less uniformly. The asymptotics established here is subsequently verified numerically in affirmative as sharp. The respective tests are conducted on 2D and 2D curves.

Keywords: Spline Interpolation, Fitting Reduced Data, Curve Modeling, Convergence Orders and Approximation, Computer Graphics

1 Introduction

Let $\gamma : [0,T] \to \mathbb{E}^n$ be a smooth regular curve (i.e. $\dot{\gamma}(t) \neq \mathbf{0}$) over $t \in [0,T]$, for $0 < T < \infty$ (see e.g. [5]). The sequence of $m+1$ interpolation points $Q_m = \{q_i\}_{i=0}^m$ in arbitrary Euclidean space $\mathbb{E}^n$ is coined as reduced data. Additionally,

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DOI: 10.1007/978-3-031-97557-8_23

it is assumed that for $q_i=\gamma(t_i)$ we have $q_{i+1}\neq q_i$. The interpolation knots $\mathcal{T}=\{t_i\}_{i=0}^m$ (with $t_i<t_{i+1}$) are here not supplied. In order to derive an interpolant $\hat{\gamma}$, first the knot estimates $\hat{\mathcal{T}}=\{\hat{t}_i\}_{i=0}^m\approx\mathcal{T}$ should be somehow guessed subject to $\hat{\gamma}(\hat{t}_i)=q_i$. Upon selecting specific interpolation scheme $\hat{\gamma}:[0,\hat{T}]\to\mathbb{E}^n$ and the substitutes $\hat{\mathcal{T}}$ of $\mathcal{T}$ a natural question arises referring to the convergence order α (if any) while estimating γ with $\hat{\gamma}$ in norm infinity. The latter stipulates $m\to\infty$ which equivalently assumes Q_m as getting dense. The desirable choice of $\{\hat{t}_i\}_{i=0}^m$ should ensure convergence of $\hat{\gamma}$ to γ with possibly a fast order α.

A background information (see e.g. [17]) is now introduced.

Definition 1. *The interpolation knots $\{t_i\}_{i=0}^m$ are called admissible if:*

$$\lim_{m\to\infty}\delta_m\to 0^+,\quad \text{where}\quad \delta_m=\max_{1\leq i\leq m}\{t_i-t_{i-1}:i=1,2,\ldots,m\}. \tag{1}$$

Recall now a special subfamily of admissible samplings i.e. the so-called *more-or-less uniform samplings* (see [35]):

Definition 2. *The sampling $\{t_i\}_{i=0}^m$ is more-or-less uniform if for some constants $0<K_l\leq K_u$ and sufficiently large m the following holds:*

$$\frac{K_l}{m}\leq t_i-t_{i-1}\leq\frac{K_u}{m}, \tag{2}$$

for all $i=1,2,\ldots,m$. Note that, condition (2) can be substituted by the inequality $\beta\delta_m\leq t_{i+1}-t_i\leq\delta_m$ holding for some $0<\beta\leq 1$ asymptotically (i.e. for sufficiently large m).

A good performance of any $\hat{\gamma}$ relies on appropriate guesses $\hat{\mathcal{T}}$ of $\mathcal{T}$. At this point recall a definition of *exponential parameterization* (see [28]):

$$\hat{t}_0^\lambda=0\quad\text{and}\quad\hat{t}_i^\lambda=\hat{t}_{i-1}^\lambda+\|q_i-q_{i-1}\|^\lambda, \tag{3}$$

for $i=1,2,\ldots,m$ and $\lambda\in[0,1]$. If $\lambda=0$, *a uniform* distribution of knots $\hat{t}_i^0=i$ eventuates. The opposite case of $\lambda=1$ is called *a cumulative chord parameterization* which yields $\hat{t}_i^1=\hat{t}_{i-1}^1+\|q_i-q_{i-1}\|$ (see [28] or [29]). Visibly the latter accounts for the geometrical dispersion of Q_m which is not reflected for $\lambda=0$ in (3). Thus, it is expected that the latter should have an impact on $\alpha(\lambda)$. Indeed, recall first:

Definition 3. *Consider a family $\{f_{\delta_m},\delta_m>0\}$ of functions $f_{\delta_m}:I\to\mathbb{E}$. We say that f_{δ_m} is of order $O(\delta_m^\alpha)$ (denoted as $f_{\delta_m}=O(\delta_m^\alpha)$), if there is a constant $K>0$ such that, for some $\bar{\delta}>0$ the inequality $|f_{\delta_m}(t)|<K\delta_m^\alpha$ holds for all $\delta_m\in(0,\bar{\delta})$, uniformly over I. For the family of vector-value functions $F_{\delta_m}:I\to\mathbb{E}^n$ by $F_{\delta_m}=O(\delta_m^\alpha)$ it is understood that $\|F_{\delta_m}\|=O(\delta_m^\alpha)$.*

Definition 4. *For a given scheme $\hat{\gamma}$ interpolating Q_m with some knots' estimates $\hat{\mathcal{T}}\approx\mathcal{T}$ (and some chosen mapping $\phi:I\to\hat{I}$) the asymptotics $\gamma-\hat{\gamma}\circ\phi=O(\delta_m^\alpha)$* is sharp *over I within the prescribed families of curves $\mathcal{J}$ and samplings $\mathcal{K}$, if there exist $\gamma\in\mathcal{J}$ and $\mathcal{T}\in\mathcal{K}$ such that for some $t^*\in I$ and some $K>0$ we have $\|\gamma(t^*)-(\hat{\gamma}\circ\phi)(t^*)\|=K\delta_m^\alpha+O(\delta_m^\theta)$, where $\theta>\alpha$.*

From now on we omit the superscript λ in $\hat{t}_i^\lambda=\hat{t}_i$ (see (3)).

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DOI: 10.1007/978-3-031-97557-8_23

2 Previous Results versus this Work Contribution

It is proved [26] that $\hat{\gamma}_3$ (i.e. a Lagrange piecewise-cubic) based on (3) and Q_m results in $\alpha(1) = 4$ (for (1)) and in $\alpha(\lambda) = 1$ for $\lambda \in [0, 1)$ (with (2)). This yields a left-hand side discontinuity of $\alpha(\lambda)$ at $\lambda = 1$ (see e.g. [20]). In fact we have:

Theorem 1. *Let γ be a regular $C^4([0,T])$ curve in $\mathbb{E}^n$ sampled more-or-less uniformly (2). Assume that $\{\hat{t}_i^\lambda\}_{i=0}^m$ are defined according to (3). Then there exists a piecewise-cubic C^∞ mapping $\psi : [0,T] \to [0,\hat{T}]$, such that over $[0,T]$:*

$$
\begin{aligned}
(\hat{\gamma}_3 \circ \psi)(t) - \gamma(t) &= O(\delta_m), \quad \textit{for} \;\; \lambda \in [0,1) \\
(\hat{\gamma}_3 \circ \psi)(t) - \gamma(t) &= O(\delta_m^4), \quad \textit{for} \;\; \lambda = 1.
\end{aligned} \tag{4}
$$

Here the mapping $\psi = \psi_3$ is defined as a piecewise-cubic Lagrange polynomial satisfying the conditions $\psi_3(t_i) = \hat{t}_i$ (for $i = 0, 1, \ldots, m$) to comply with the interpolation constraints $q_i = \hat{\gamma}(\hat{t}_i) = \hat{\gamma}(\psi(t_i))$.

Noticeably, the continuous interpolant $\hat{\gamma}_3$ is generically non-smooth at junction points $\{q_k\}_{k=1}^{3k}$, i.e. where two consecutive local piecewise-cubics are glued together.

One option is to consider any C^1 interpolation scheme based on extra provision of the unknown velocities $\{\boldsymbol{v}_i\}_{i=0}^m$ at Q_m. A possible remedy is proposed in [18] and [24], where a modified Hermite C^1 fitting scheme $\hat{\gamma}_H$ is introduced and analyzed in conjunction with (3). In particular, the asymptotics established for γ_H based on (3) coincides with (4) from Th. 1.

The solution guaranteeing C^2 smoothness at Q_m resorts to various hybrids of C^2 cubic spline interpolants $\hat{\gamma}_3^S$ (see [4]) based on Q_m and (3).

First of them called *a complete cubic spline* $\hat{\gamma}_3^C$ (see [4]) requires an initial $\boldsymbol{v}_0 = \gamma'(0)$ and terminal $\boldsymbol{v}_n = \gamma'(T)$ velocities, generically not accompanying reduced data Q_m. This special case is discussed in [10] (though limited exclusively to $\lambda = 1$), where quartic order $\alpha(1) = 4$ for trajectory estimation by $\hat{\gamma}_3^C$ is established.

A possible alternative of C^2 class fitting scheme based merely on Q_m and (3) (with no reference to $\boldsymbol{v}_0$ and $\boldsymbol{v}_m$) is e.g. given in [25]. The latter introduces *a modified complete spline* $\hat{\gamma}_3^{MC}$, where both initial and terminal velocities are estimated by the derivatives of the first and the last components of the Lagrange piecewise-cubic $\hat{\gamma}_3$ (for which $\hat{\gamma}_{3,0}'(0) \approx \boldsymbol{v}_0$ and $\hat{\gamma}_{3,m-3}'(T) \approx \boldsymbol{v}_m$)). More details concerning the construction of $\hat{\gamma}_3^{MC}$ are given next in Section 3.

The *main contribution of this paper* is to establish *a linear convergence order* in γ estimation with *a modified complete spline* $\hat{\gamma}_3^{MC}$ based on exponential parameterization (3), for $\lambda \in [0, 1)$ - *see Th. 2.* The latter holds for any regular curve $\gamma \in C^4$ sampled more-or-less uniformly (2) *on dense reduced data* Q_m. Here the analysis assumes $m \to \infty$ assuring a good approximation property in $\hat{\gamma} \approx \gamma$ for m getting large. *Numerical tests* are also performed on 2D and 3D regular curves to confirm the *asymptotics together with its sharpness* determined by (13) and (14).

Related work on *fitting sparse or dense reduced data* with optimal knots selection criterion based on using C^2 class splines (see Section 3) can also be

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DOI: 10.1007/978-3-031-97557-8_23

found e.g. in [21], [22], [23] or [27]. Other curve interpolation schemes combined with various parameterizations (with some specific applications given) are also studied e.g. in [2], [6], [7], [11], [12], [14], [31], [38] or [39].

3 Spline Construction

The construction of *a modified complete spline interpolant* $\hat{\gamma}_3^{MC}$ based on reduced data Q_m (see also [4]) and exponential parameterization (3) falls into the following steps (a similar procedure renders γ_3^{MC} for non-reduced data $(\mathcal{T}, Q_m)$):

1. Calculate the estimates $\{\hat{t}_i\}_{i=0}^m$ of the missing knots $\{t_i\}_{i=0}^m$ according to the exponential parameterization (3) (with $\lambda \in [0,1]$).
2. The so-called general C^2 piecewise-cubic spline $\hat{\gamma}_3^S$ interpolant (a sum-track of cubics $\{\hat{\gamma}_{3,i}^S\}_{i=0}^{m-1}$ - see [4]) fulfills the following condition (over each subsegment $\hat{I}_i = [\hat{t}_i, \hat{t}_{i+1}]$):
$$\hat{\gamma}_{3,i}^S(\hat{t}_i) = q_i, \qquad \hat{\gamma}_{3,i}^S(\hat{t}_{i+1}) = q_{i+1},$$
$$\hat{\gamma}_{3,i}^{S'}(\hat{t}_i) = \boldsymbol{v}_i, \qquad \hat{\gamma}_{3,i}^{S'}(\hat{t}_{i+1}) = \boldsymbol{v}_{i+1}, \tag{5}$$
where $\boldsymbol{v}_0, \cdots, \boldsymbol{v}_m$ represent the unknown slopes (i.e. velocities) $\boldsymbol{v}_i \in \mathbb{R}^n$. The internal velocities $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{m-1}\}$ must satisfy C^2 class $m-1$ constraints imposed on $\hat{\gamma}_3^S$ at junction points $\{q_1, \ldots, q_{m-1}\}$ i.e. by enforcing:
$$\hat{\gamma}_{3,i-1}^{S''}(\hat{t}_i) = \hat{\gamma}_{3,i}^{S''}(\hat{t}_i). \tag{6}$$
They can be uniquely computed (see [4] or (9) and Section 4) provided both $\boldsymbol{v}_0$ and $\boldsymbol{v}_m$ are somehow known (or a priori given).
3. Assuming temporarily the provision of all velocities $\{\boldsymbol{v}_i\}_{i=0}^m$, each cubic $\hat{\gamma}_{3,i}^S$ over $\hat{t} \in [\hat{t}_i, \hat{t}_{i+1}]$ reads as:
$$\hat{\gamma}_{3,i}^S(\hat{t}) = c_{1,i} + c_{2,i}(\hat{t} - \hat{t}_i) + c_{3,i}(\hat{t} - \hat{t}_i)^2 + c_{4,i}(\hat{t} - \hat{t}_i)^3, \tag{7}$$
where its respective coefficients (with $\Delta\hat{t}_i = \hat{t}_{i+1} - \hat{t}_i$) are equal to:
$$c_{1,i} = q_i, \quad c_{2,i} = \boldsymbol{v}_i,$$
$$c_{3,i} = \frac{\frac{q_{i+1}-q_i}{\Delta\hat{t}_i} - \boldsymbol{v}_i}{\Delta\hat{t}_i} - c_{4,i}\Delta\hat{t}_i, \quad c_{4,i} = \frac{\boldsymbol{v}_i + \boldsymbol{v}_{i+1} - 2\frac{q_{i+1}-q_i}{\Delta\hat{t}_i}}{(\Delta\hat{t}_i)^2}. \tag{8}$$
If also $\boldsymbol{v}_i = \gamma'(t_i)$ are given then formulas (7) and (8) yield a well-known C^1 class Hermite spline. However, the required velocities $\{\boldsymbol{v}_0, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_m\}$ are not usually supplemented to Q_m. A scheme for computing the corresponding missing internal velocities $\{\boldsymbol{v}_1, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_{m-1}\}$ is recalled next (see [4]). Extending the latter a method of estimating $\{\boldsymbol{v}_0, \boldsymbol{v}_m\}$ is given in [18] - see below for more details.

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DOI: 10.1007/978-3-031-97557-8_23

4. Formulas (7) and (8) render $\hat{\gamma}_{3,i}^{S''}(\hat{t}_i) = 2c_{3,i}$ and $\hat{\gamma}_{3,i-1}^{S''}(\hat{t}_i) = 2c_{3,i-1} + 6c_{4,i-1}(\hat{t}_i - \hat{t}_{i-1})$ which combined with (6) leads to the linear system (for $i = 1, 2 \ldots m-1$):

$$\boldsymbol{v}_{i-1}\Delta\hat{t}_i + 2\boldsymbol{v}_i(\Delta\hat{t}_{i-1} + \Delta\hat{t}_i) + \boldsymbol{v}_{i+1}\Delta\hat{t}_{i-1} = b_i, \tag{9}$$

where

$$b_i = 3\left(\Delta\hat{t}_i \frac{q_i - q_{i-1}}{\Delta\hat{t}_{i-1}} + \Delta\hat{t}_{i-1}\frac{q_{i+1} - q_i}{\Delta\hat{t}_i}\right). \tag{10}$$

Assuming that the end-slopes $\boldsymbol{v}_0$ and $\boldsymbol{v}_m$ are somehow given the tridiagonal system (9) solves uniquely in $\{\boldsymbol{v}_i\}_{i=1}^{m-1}$ - see [4]. The latter yields a C^2 spline $\hat{\gamma}_3^S$ (which fits reduced data Q_m) defined as a track-sum of $\{\hat{\gamma}_{3,i}^S\}_{i=0}^{m-1}$ introduced in (7). If extra conditions hold, i.e. $\gamma'(t_0) = \boldsymbol{v}_0$ and $\gamma'(T) = \boldsymbol{v}_m$ then $\hat{\gamma}_3^S$ is called *a complete cubic spline* (denoted here as $\hat{\gamma}_3^{CS}$).
5. Since Q_m are usually deprived from both initial and terminal velocities $\{\gamma'(t_0) = \boldsymbol{v}_0, \gamma'(T) = \boldsymbol{v}_m\}$ a good estimate $\{\boldsymbol{v}_0^a, \boldsymbol{v}_m^a\}$ is therefore required. Of course, any choice of $\{\boldsymbol{v}_0^a, \boldsymbol{v}_m^a\}$ renders a unique explicit formula for modification of $\hat{\gamma}_3^{CS}$. This however is insufficient for our consideration. Indeed to preserve a good approximation property of $\hat{\gamma}$, still a good estimate of these two velocities is required so that (13) and (14) hold. In doing so, we apply Lagrange cubic $\hat{\gamma}_{3,0}^L : [0, \hat{t}_3] \to \mathbb{E}^n$ (and $\hat{\gamma}_{3,m-3}^L : [\hat{t}_{m-3}, \hat{T}] \to \mathbb{E}^n$), satisfying $\hat{\gamma}_{3,0}^L(\hat{t}_i) = q_i$ (and $\hat{\gamma}_{3,m-3}^L(\hat{t}_{m-3+i}) = q_{m-3+i}$), with $i = 0, 1, 2, 3$ - here the same $\lambda \in [0,1]$ is applied in the derivation of $\hat{\gamma}_{3,0}^L$, $\hat{\gamma}_{3,m-3}^L$. With such velocities the resulting complete spline $\hat{\gamma}_3^C$ is called *a modified complete spline* (denoted as $\hat{\gamma}_3^{MC}$) for which $\boldsymbol{v}_0^a = \hat{\gamma}_{3,0}^{L'}(0)$ and $\boldsymbol{v}_m^a = \hat{\gamma}_{3,m-3}^{L'}(\hat{T})$.
6. However, to verify the asymptotics from (13) and (14) a candidate for a mapping $\psi : [0, T] \to [0, \hat{T}]$ is still required. In doing so, consider a C^2 complete spline $\psi = \psi_3^C : [0, T] \to [0, \hat{T}]$ satisfying the knots' interpolation constraints $\psi_3^C(t_i) = \hat{t}_i$, where $\{\hat{t}_i\}_{i=0}^m$ are defined according to (3) (in principle this procedure extends to any $\hat{\mathcal{T}}$). In addition, the initial and terminal velocities of $s_0 = \psi_3^{C'}(0)$ and $s_m = \psi_3^{C'}(T)$ are set similarly to the construction from above. The internal velocities $\{s_i\}_{i=1}^{m-1}$ (defined by $s_i = \dot{\psi}_3^C(t_i)$) satisfy the analogous constraints to those from (9) and (10) (for $i = 1, 2, \ldots m-1$):

$$s_{i-1}\Delta t_i + 2(\Delta t_{i-1} + \Delta t_i)s_i + s_{i+1}\Delta t_{i-1} = a_i, \tag{11}$$

where

$$a_i = 3\left(\Delta t_i \frac{\hat{t}_i - \hat{t}_{i-1}}{\Delta t_{i-1}} + \Delta t_{i-1}\frac{\hat{t}_{i+1} - \hat{t}_i}{\Delta t_i}\right). \tag{12}$$

To generate both estimate of s_0 and s_m, define two Lagrange cubics $\psi_{3,0} : [0, t_3] \to [0, \hat{t}_3]$ and $\psi_{3,m-3} : [t_{m-3}, T] \to [\hat{t}_{m-3}, \hat{T}]$ satisfying interpolation conditions $\psi_{3,0}(t_i) = \hat{t}_i$ and $\psi_{3,m-3}(t_{m-3+i}) = \hat{t}_{m-3+i}$ (with $i = 0, 1, 2, 3$ and the same $\lambda \in [0,1]$ as for the construction of $\hat{\gamma}_3^C$), respectively. In sequel, one approximates here $s_0 = \psi_3^{C'}(0)$ with $\psi'_{3,0}(0)$ and $s_m = \psi_3^{C'}(T)$ with $\psi'_{3,m-3}(T)$. Such spline ψ_3^C is also called a modified complete spline and is analogously denoted here by ψ_3^{MC}.

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DOI: 10.1007/978-3-031-97557-8_23

This completes *a construction of a modified C^2 complete spline* $\hat{\gamma}_3^{MC}$ (and of $\hat{\psi}_3^{MC}$)$\hat{\gamma}_3^{MC}$ (and of $\hat{\psi}_3^{MC}$) based on reduced data Q_m and exponential parameterization (3). Noticeably, with m increasing the terminal velocities for $\hat{\gamma}_3^{MC}$ and $\hat{\psi}_3^{MC}$ must be re-estimated for each m in accordance with the procedure specified above.

Note that if $[\psi_i^{MC}(t_i) = \hat{t}_i, \psi_i^{MC}(t_{i+1}) = \hat{t}_{i+1}] \subsetneq \psi_{3,i}^{MC}([t_i, t_{i+1}])$ then one has to extend the domain of $\hat{\gamma}_{3,i}^{MC}$ from $[\hat{t}_i, \hat{t}_{i+1}]$ to $\mathbb{R}$ to enable calculation $\hat{\gamma}_{3,i}^{MC} \circ \psi_{3,i}^{MC}$. Such $\hat{\gamma}_{3,i}^{MC}$ is denoted by $\tilde{\gamma}_{3,i}^{MC}$ which obviously satisfies $\tilde{\gamma}_{3,i}^{MC}|_{[\hat{t}_i, \hat{t}_{i+1}]} = \hat{\gamma}_{3,i}^{MC}$. In fact the asymptotics established in Th. 2 applies to the "extended version" $\tilde{\gamma}_{3,i}^{MC} \circ \psi_{3,i}^{MC}$ of $\hat{\gamma}_{3,i}^{MC} \circ \psi_{3,i}^{MC}$ over each I_i.

4 Main Result

We establish now the main contribution of this work. The following holds:

Theorem 2. *Let γ be a regular $C^4([0,T])$ curve in $\mathbb{E}^n$ sampled more-or-less-uniformly* (3). *Let $\boldsymbol{v}_0^a = \hat{\gamma}_3^{L'}(0)$ and $\boldsymbol{v}_m^a = \hat{\gamma}_3^{L'}(\hat{T})$, where $\hat{\gamma}_3^L$ defines a piecewise-cubic Lagrange based on Q_m and* (3) *with $\lambda \in [0,1]$. Assume also that $\hat{\gamma}_3^{MC}$: $[0,\hat{T}] \to \mathbb{E}^n$ define a modified complete spline based on Q_m, ($\boldsymbol{v}_0^a$, $\boldsymbol{v}_m^a$) and* (3). *Then there is a piecewise-C^∞ mapping $\psi = \psi_3^{MC} : [0,T] \to [0,\hat{T}]$ such that over $[0,T]$ we either have for all $\lambda \in [0,1)$:*

$$\tilde{\gamma}_3^{MC} \circ \psi - \gamma = O(\delta_m) \tag{13}$$

or for $\lambda = 1$:

$$\tilde{\gamma}_3^{MC} \circ \psi - \gamma = O(\delta_m^4). \tag{14}$$

Proof. Taking into account that velocities $\boldsymbol{v}_0$, $\boldsymbol{v}_m$, s_0 and s_m are estimated (see Section 3) both (9) (with (10)) and (11) (with (12)) represent two quadratic tridiagonal linear systems of $m-2$ equations (each in $m-2$ unknowns) which are strictly row diagonally dominant. Thus each system has exactly one solution which can be found e.g. by Gauss elimination without pivoting. The following inequalities hold (see [4], Chap. 4, Problem 7):

$$\max_{0\le i\le m} \|\boldsymbol{v}_i\| \le \max\{\|\boldsymbol{v}_0\|, \max_{1\le j\le m-1} \frac{\|b_j\|}{\Delta\hat{t}_{j-1} + \Delta\hat{t}_j}, \|\boldsymbol{v}_m\|\} \tag{15}$$

and

$$\max_{0\le i\le m} |s_i| \le \max\{|s_0|, \max_{1\le j\le m-1} \frac{|a_j|}{\Delta t_{j-1} + \Delta t_j}, |s_m|\}. \tag{16}$$

The proof of Th. 2 is performed here only for $\lambda \in [0,1)$ in (3). The case of $\lambda = 1$ exceeds the scope of this paper. By [26] (one assumes here $\gamma \in C^4$ sampled more-or-less uniformly along (2)), each pair of initial and terminal velocities satisfies (for $k = 0, m$):

$$\boldsymbol{v}_k = O(\delta_m^{1-\lambda}) \quad \text{and} \quad s_k = O(\delta_m^{\lambda-1}),$$

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thus yielding the following asymptotics (for $k = 0, m$):

$$\|\boldsymbol{v}_k\| = O(\delta_m^{1-\lambda}) \quad \text{and} \quad |s_k| = O(\delta_m^{\lambda-1}). \tag{17}$$

In order to determine the asymptotics from the right-hand side of (15) (and of (16)) the remaining middle terms are now examined. Substituting (10) into (15) (and (12) into (16)) renders two expressions (for $j = 1, \ldots, m-1$):

$$I_{\boldsymbol{v}} = 3 \left\| \frac{\Delta\hat{t}_j \frac{q_j - q_{j-1}}{\Delta\hat{t}_{j-1}} + \Delta\hat{t}_{j-1} \frac{q_{j+1} - q_j}{\Delta\hat{t}_j}}{\Delta\hat{t}_{j-1} + \Delta\hat{t}_j} \right\|, \quad I_s = 3 \left| \frac{\Delta t_j \frac{\hat{t}_j - \hat{t}_{j-1}}{\Delta t_{j-1}} + \Delta t_{j-1} \frac{\hat{t}_{j+1} - \hat{t}_j}{\Delta t_j}}{\Delta t_{j-1} + \Delta t_j} \right| \tag{18}$$

which asymptotics needs further analysis. In doing so, recall that the curve γ as a regular curve can be parameterized by arc-length parameterization (see e.g. [5] or [15]) yielding $\|\dot{\gamma}\| = 1$. Hence upon differentiating both sides $\|\dot{\gamma}(t)\|^2 = \langle\dot{\gamma}(t)|\dot{\gamma}(t)\rangle = 1$ the orthogonality condition $\langle\dot{\gamma}(t)|\ddot{\gamma}(t)\rangle = 0$ follows. Consequently Taylor expansion applied to γ renders the following (as $\|\boldsymbol{w}\|^2 = \langle\boldsymbol{w}|\boldsymbol{w}\rangle$):

$$\begin{aligned}
\hat{t}_{j+1} - \hat{t}_j &= \|\gamma(t_{j+1}) - \gamma(t_j)\|^\lambda \\
&= (t_{j+1} - t_j)^\lambda)\|\dot{\gamma}(t_j) + \frac{(t_{j+1} - t_j)}{2}\ddot{\gamma}(t_j) + O((t_{j+1} - t_j)^2)\|^\lambda \\
&= (t_{j+1} - t_j)^\lambda) \left(\|\dot{\gamma}(t_j) + \frac{(t_{j+1} - t_j)}{2}\ddot{\gamma}(t_j) + O((t_{j+1} - t_j)^2)\|^2 \right)^{\frac{\lambda}{2}} \\
&= (t_{j+1} - t_j)^\lambda [1 + O((t_{j+1} - t_j)^2]^{\frac{\lambda}{2}}.
\end{aligned} \tag{19}$$

Again Taylor expansion of $f(y) = (1+y)^{\frac{\lambda}{2}}$ yields $f(y) = 1 + \frac{\lambda}{2}(1+\xi)^{\frac{\lambda}{2}-1}y$ for some $\xi \in [0, y]$ or $\xi \in [0, y]$. Thus for such ξ (if y is bounded) the expression $\frac{\lambda}{2}(1+\xi)^{\frac{\lambda}{2}-1} = O(1)$ and therefore $f(y) = 1 + O(y)$. Substituting for $y = O((t_{j+1} - t_j)^2)$ in the latter together with (19) results in:

$$\hat{t}_{j+1} - \hat{t}_j = (t_{j+1} - t_j)^\lambda (1 + O((t_{j+1} - t_j)^2) = (t_{j+1} - t_j)^\lambda + O((t_{j+1} - t_j)^{2+\lambda}). \tag{20}$$

Combining the latter with Taylor expansion of γ leads to: $(q_{j+1} - q_j)/\Delta\hat{t}_j$

$$\begin{aligned}
&= \frac{\gamma(t_{j+1}) - \gamma(t_j)}{\hat{t}_{j+1} - \hat{t}_j} = \frac{\gamma(t_{j+1}) - \gamma(t_j)}{\|\gamma(t_{j+1}) - \gamma(t_j)\|^\lambda} \\
&= \frac{(t_{j+1} - t_j)[\dot{\gamma}(t_j) + \frac{(t_{j+1} - t_j)}{2}\ddot{\gamma}(t_j) + O((t_{j+1} - t_j)^2)]}{(t_{j+1} - t_j)^\lambda [1 + O((t_{j+1} - t_j)^2)]} \\
&= (t_{j+1} - t_j)^{1-\lambda} [\dot{\gamma}(t_j) + \frac{(t_{j+1} - t_j)}{2}\ddot{\gamma}(t_j) + O((t_{j+1} - t_j)^2)][1 + O((t_{j+1} - t_j)^2)] \\
&= (t_{j+1} - t_j)^{1-\lambda} (O(1) + O((t_{j+1} - t_j)^2) + O((t_{j+1} - t_j)^4) \\
&= O(\delta_m^{1-\lambda}) + O(\delta_m^{3-\lambda}) = O(\delta_m^{1-\lambda}).
\end{aligned} \tag{21}$$

Analogously one arrives at

$$\frac{q_j - q_{j-1}}{\Delta\hat{t}_{j-1}} = O(\delta_m^{1-\lambda}). \tag{22}$$

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Coupling $\Delta\hat{t}_{j-k}/(\Delta\hat{t}_{j-1}+\Delta\hat{t}_j)=O(1)$ (for $k=0,1$) with (21) and (22) renders the asymptotics of the first formula from (18) as $I_{\boldsymbol{v}}=O(\delta_m^{1-\lambda})$. The latter together with (15) and (17) yields (for all $i=0,1,2,\ldots,m$):

$$\|\boldsymbol{v}_i\|=O(\delta_m^{1-\lambda}). \tag{23}$$

Similarly, by (20) the following holds (by more-or-less uniformity of $\mathcal{T}$):

$$\frac{\hat{t}_{j+1}-\hat{t}_j}{\Delta t_j}=(t_{j+1}-t_j)^{\lambda-1}(1+O((t_{j+1}-t_j)^2)=O(\delta_m^{\lambda-1}). \tag{24}$$

Analogously one obtains (again by more-or-less uniformity of $\mathcal{T}$):

$$\frac{\hat{t}_j-\hat{t}_{j-1}}{\Delta t_{j-1}}=(t_j-t_{j-1})^{\lambda-1}(1+O((t_j-t_{j-1})^2)=O(\delta_m^{\lambda-1}). \tag{25}$$

As previously, coupling $\Delta t_{j-k}/(\Delta t_{j-1}+\Delta t_j)=O(1)$ (for $k=0,1$) together with (24) and (25) guarantees the second formula in (18) as $I_s=O(\delta_m^{\lambda-1})$. Hence by (16) and (17) the following holds (for all $i=0,1,2,\ldots,m$):

$$|s_i|=O(\delta_m^{\lambda-1}). \tag{26}$$

We are ready now to determine the asymptotics of the expression $f(t)=(\check{\gamma}_3^{MC}\circ\psi_3^{MC})(t)-\gamma(t)$ over $[0,T]$, which permits to establish the order in γ estimation by $\check{\gamma}_3^{MC}\circ\psi_3^{MC}$. Evidently, in doing so, it suffices to examine the latter over each sub-segment $I_i=[t_i,t_{i+1}]$ i.e. for each $f_i(t)=(\check{\gamma}_{3,i}^{MC}\circ\psi_{3,i}^{MC})(t)-\gamma(t)$. From now on, to abbreviate the notation shorter symbols $\check{\gamma}_{3,i}=\check{\gamma}_{3,i}^{MC}$ and $\psi_{3,i}=\psi_{3,i}^{MC}$ are used. Since $f_i(t_{i+k})=\mathbf{0}$ (for $k=0,1$) by Hadamard's Lemma [33] and chain rule one arrives at (for $t\in I_i$):

$$f_i(t)=(t-t_i)(t-t_{i+1})O(\ddot{f}_i)=(t-t_1)(t-t_{i+1})O\left(\check{\gamma}_{3,i}''\dot{\psi}_{3,i}^2+\check{\gamma}_{3,i}'\ddot{\psi}_{3,i}-\ddot{\gamma}\right). \tag{27}$$

Newton Interpolation formula [4] leads to:

$$\begin{aligned}\dot{\psi}_{3,i}(t)=\psi_{3,i}[t_i,t_i]&+2\psi_{3,i}[t_i,t_i,t_{i+1}](t-t_i)\\&+\psi_{3,i}[t_i,t_i,t_{i+1},t_{i+1}](2(t-t_i)(t-t_{i+1})+(t-t_i)^2).\end{aligned} \tag{28}$$

Upon combining (2), (24), (25) with (26) one obtains:

$$\begin{aligned}\psi_{3,i}[t_i,t_i]&=s_i=O(\delta_m^{\lambda-1}),\\ \psi_{3,i}[t_i,t_i,t_{i+1}](t-t_i)&=(\psi_{3,i}[t_i,t_{i+1}]-s_i)\frac{t-t_i}{t_{i+1}-t_i}=(O(\delta_m^{\lambda-1})+O(\delta_m^{\lambda-1}))O(1)\\&=O(\delta_m^{\lambda-1}),\\ \psi_{3,i}[t_i,t_i,t_{i+1},t_{i+1}]&=\frac{s_{i+1}-\psi_{i3}[t_i,t_{i+1}]}{(t_{i+1}-t_i)^2}-\frac{\psi_{3,i}[t_i,t_i,t_{i+1}](t_{i+1}-t_i)}{(t_{i+1}-t_i)^2}=O(\delta_m^{\lambda-3}).\end{aligned} \tag{29}$$

Substituting now (29) into (28) renders:

$$\dot{\psi}_{3,i}(t)=O(\delta_m^{\lambda-1})\quad\text{and}\quad\dot{\psi}_{3,i}^2(t)=O(\delta_m^{2\lambda-2}). \tag{30}$$

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A simple inspection combined with (29) leads to:

$$\begin{aligned}\ddot{\psi}_{3,i}(t) &= 2\psi_{3,i}[t_i,t_i,t_{i+1}] + 2\psi_{3,i}[t_i,t_i,t_{i+1},t_{i+1}](2(t-t_i)+(t-t_{i+1}))\\ &= O(\delta_m^{\lambda-2}) + O(\delta_m^{\lambda-2}) = O(\delta_m^{\lambda-2}).\end{aligned} \tag{31}$$

In the next step, the asymptotics $\check{\gamma}'_{3,i}$ and $\check{\gamma}''_{3,i}$ is investigated. In doing so, Newton interpolation formula [4] applied to $\check{\gamma}_{3,i}$ yields:

$$\begin{aligned}\check{\gamma}'_{3,i}(\hat{t}) = \check{\gamma}_{3,i}[\hat{t}_i,\hat{t}_i] + 2\check{\gamma}_{3,i}[\hat{t}_i,\hat{t}_i,\hat{t}_{i+1}](\hat{t}-\hat{t}_i)\\ +\check{\gamma}_{3,i}[\hat{t}_i,\hat{t}_i,\hat{t}_{i+1},\hat{t}_{i+1}](2(\hat{t}-\hat{t}_i)(\hat{t}-\hat{t}_{i+1})+(\hat{t}-\hat{t}_i)^2).\end{aligned} \tag{32}$$

Coupling together (20), (21) and (23) renders:

$$\begin{aligned}\check{\gamma}_{3,i}[\hat{t}_i,\hat{t}_i] &= \boldsymbol{v}_i = O(\delta_m^{1-\lambda}),\\ \check{\gamma}_{3,i}[\hat{t}_i,\hat{t}_i,\hat{t}_{i+1}](\hat{t}-\hat{t}_i) &= \frac{\check{\gamma}_{3,i}[\hat{t}_i,\hat{t}_{i+1}]-\boldsymbol{v}_i}{\hat{t}_{i+1}-\hat{t}_i}(\hat{t}-\hat{t}_i)\\ &= \frac{O(\delta_m^{1-2\lambda})}{1+O((t_{i+1}-t_i)^2)}(\hat{t}-\hat{t}_i) = O(\delta_m^{1-\lambda}).\end{aligned} \tag{33}$$

To justify the last step in (33), note that Mean Value Th. with (30) (for all $t \in I_i$) yield $\hat{t}-\hat{t}_i = \frac{\psi_{3,i}(t)-\psi_{3,i}(t_i)}{t-t_i}(t-t_i) = \dot{\psi}_{3,i}(\xi)(t-t_i) = O(\delta_m^{\lambda-1})\cdot O(\delta_m) = O(\delta_m^{\lambda})$. Additionally, Taylor expansion applied to $f(x) = (1+x^2)^{-1}$ renders $(1+O(\delta_m^2))^{-1} = 1+O(\delta_m^2)$. A similar argument as used in (33) (see also (29)) assures the following (for $t \in I_i$):

$$\check{\gamma}_{3,i}[\hat{t}_i,\hat{t}_i,\hat{t}_{i+1},\hat{t}_{i+1}](2(\hat{t}-\hat{t}_i)(\hat{t}-\hat{t}_{i+1})+(\hat{t}-\hat{t}_i)^2) = O(\delta_m^{1-3\lambda})O(\delta_m^{2\lambda}) = O(\delta_m^{1-\lambda}). \tag{34}$$

Consequently, both (33) with (34) result in the asymptotics:

$$\check{\gamma}'_{3,i}(\hat{t}) = O(\delta_m^{1-\lambda}) \quad \text{and} \quad \check{\gamma}''_{3,i}(\hat{t}) = O(\delta_m^{1-2\lambda}). \tag{35}$$

Finally, substituting (30), (31) and (35) into (27) renders the following asymptotics (over each I_i and $\lambda \in [0,1)$):

$$\begin{aligned}f_i(t) &= O(\delta_m^2)\big(O(\delta_m^{1-2\lambda})O(\delta_m^{2\lambda-2}) + O(\delta_m^{1-\lambda})O(\delta_m^{\lambda-2}) + O(1)\big)\\ &= O(\delta_m) + O(\delta_m^2) = O(\delta_m).\end{aligned} \tag{36}$$

This completes the proof of (13) in Th. 2. The case of $\lambda = 1$ rendering a quartic convergence order (14) is here omitted. Note that in (36) the term $(t-t_i)(t-t_{i+1})\ddot{\gamma}(t) = O(\delta_m^2)$ forms the intrinsic quadratic barrier annihilating any improvement of the asymptotics in $O(\check{\gamma}''_{3,i}\dot{\psi}_{3,i}^2 + \check{\gamma}'_{3,i}\ddot{\psi}_{3,i})$ beyond $O(\delta_m^2)$. Thus the argument used herein prevails only for $\lambda \in [0,1)$ and as such needs modification for $\lambda = 1$. Alternatively, recall that $\lambda = 1$ in (3) is analyzed in [10] for complete spline $\hat{\gamma}_3^{CS}$ only, i.e. with $\boldsymbol{v}_0$ and $\boldsymbol{v}_m$ a priori given. The adaptation of the latter to $\hat{\gamma}_3^{MC}$ forms an alternative tool to justify (14). □

The next section reports on numerical testing confirming the asymptotics together with its sharpness established in Th. 2.

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5 Experiments

In this section, a numerical verification of the asymptotics $\alpha(\lambda)$ (and its sharpness) from Th. 2 is conducted. Recall that, given fixed $\lambda \in [0,1]$, by sharpness (see Def. 4) we understand the existence of at least one curve $\gamma \in C^4(0,T])$ and one special family $\mathcal{T}$ of more-or-less uniform sampling (2) such that the asymptotics $O(\delta_m^{\alpha(\lambda)})$ in difference $\tilde{\gamma}_3^{MC} \circ \psi_3^{MC} - \gamma$ (over $[0,T]$) is not faster than $\alpha(\lambda)$. A confirmation of (13) and (14) indicates again on an unexpected left-hand side discontinuity in $\alpha(\lambda)$ at $\lambda = 1$.

All tests are performed in *Mathematica 12.0* and use to two types of skew-symmetric more-or-less uniform samplings. The first one (for $t_i \in [0,1]$) is defined as follows:

$$t_i = \begin{cases} \frac{i}{m} + \frac{1}{2m}, & \text{for } i = 4k+1; \\ \frac{i}{m} - \frac{1}{2m}, & \text{for } i = 4k+3; \\ \frac{i}{m}, & \text{for } i \text{ even}; \end{cases} \tag{37}$$

with $K_l = (1/2)$ and $K_u = (3/2)$ introduced in (2). The second sampling reads as:

$$t_i = \frac{i}{m} + \frac{(-1)^{i+1}}{3m}, \tag{38}$$

with constants $K_l = (1/2)$ and $K_u = (5/3)$ from (2). For a given m, the error E_m, between γ and reparameterized spline $\tilde{\gamma}_3^{MC} \circ \psi_3^{MC}$ is determined by the formula:

$$E_m = \max_{t \in [0,1]} \|(\tilde{\gamma}_3^{MC} \circ \psi_3^{MC})(t) - \gamma(t)\|.$$

The latter is computed over each sub-interval $[t_i, t_{i+1}]$ (for $i = 0, \cdots, m-1$) by using *Mathematica* function - *FindMaximum* and then upon taking the maximal values from all segments' optima. In order to approximate $\alpha(\lambda)$ we calculate first E_m for $m_{min} \leq m \leq m_{max}$, where m_{min} and m_{max} are sufficiently large fixed constants. Then a linear regression yielding a function $y(x) = \bar{\alpha}(\lambda)x + b$ is applied to $\{(\log(m), -\log(E_m))\}_{m_{\min}}^{m_{\max}}$. Mathematica built-in function *Linear-ModelFit* extracts a coefficient $\bar{\alpha}(\lambda) \approx \alpha(\lambda)$. A full justification of this procedure to approximate $\alpha(\lambda)$ by $\bar{\alpha}(\lambda)$ is given in [17]. Note also that since both (13) and (14) have asymptotic character the constants $m_{min} < m_{max}$ should be taken as sufficiently large. On other hand, a potential negative impact of machine rounding-off errors stipulates these two constants not to exceed big values. In practice, the appropriate choices for $m_{min} < m_{max}$ are adjusted each time during the experimental phase. The tests conducted here employ three types of C^∞ regular curves: *an epitrochoid* γ_{ep} in $\mathbb{E}^2$ (i.e. planar a curve) and two curves *a conical spiral* γ_{cs} and *a quadratic helix* γ_{qh} both in $\mathbb{E}^3$ (i.e. 3D curves). All tested curves are sampled more-or-less uniformly (3) according to either (37) or (38).

Example 1. Consider *a regular planar epitrochoid* $\gamma_{ep} : [0,1] \to \mathbb{E}^2$,

$$\gamma_{ep}(t) = (4\cos(t) - 0.15\cos(4\pi t), 4\sin(t) - 0.15\sin(4\pi t)). \tag{39}$$

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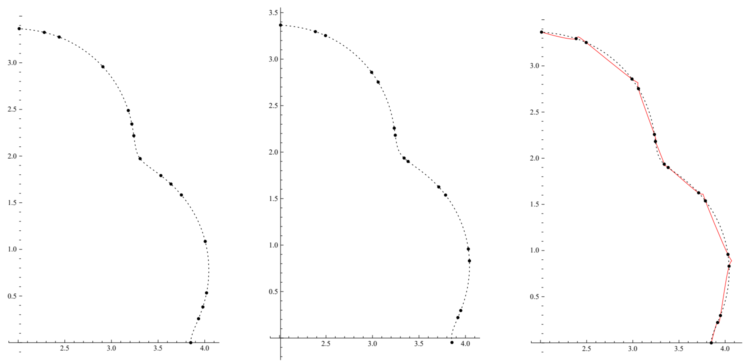

a) b) c)

Fig. 1. *An epitrochoid* γ_{ep} (39) sampled along (dotted): a) (37) or b) (38) and c) fitted $\hat{\gamma}^{MC}$ with (38) & $\lambda = 0$ for $m = 15$.

Fig. 1(a) (or Fig. 1(b)) contains the plots of γ_{ep} sampled (here $m = 15$) according to either (37) (or (38)).

The respective linear regression based estimates $\bar{\alpha}(\lambda) \approx \alpha(\lambda)$ (for various $\lambda \in [0, 1]$) are computed here for $m_{min} = 60 \leq m \leq m_{max} = 120$. The numerical results contained in Table 1 confirm the sharpness of (13) and (14) for $\lambda \in \{0.0, 0.1, 0.3, 0.5, 0.7\}$ and yield marginally faster (though still consistent with asymptotics from Th. (2)) $\alpha(\lambda)$ for $\lambda \approx 1$. Note that for $\lambda = 1$ we have $m_{min} = 240 \leq m \leq m_{max} = 270$.

Table 1. Computed $\bar{\alpha}(\lambda) \approx \alpha(\lambda)$ in (13) & (14) for γ_{ep} from (39) and various $\lambda \in [0, 1]$.

λ	0.0	0.1	0.3	0.5	0.7	0.9	1.0
$\bar{\alpha}(\lambda)$ for (37)	1.007	1.013	1.028	1.055	1.116	1.377	4.274
$\bar{\alpha}(\lambda)$ for (38)	1.037	1.036	1.042	1.066	1.143	1.483	4.259
$\alpha(\lambda)$ in Th. 2	1.0	1.0	1.0	1.0	1.0	1.0	4.0

We pass now to the example with *a quadratic helix* in $\mathbb{E}^3$.

Example 2. Let *a quadratic helix* $\gamma_{qh} : [0, 1] \to \mathbb{E}^3$ be defined as:

$$\gamma_{qh}(t) = (1.5\cos(2\pi t), \frac{2\pi t}{4}\sin(2\pi t), t). \qquad (40)$$

Again Fig. 2(a) (or Fig. 2(b)) illustrates the trajectories of γ_{qh} sampled according to either (37) or (38), with $m = 15$.

As previously, a linear regression estimating $\bar{\alpha}(\lambda) \approx \alpha(\lambda)$ from Th. 2 is used here, for m ranging between $100 \leq m \leq 160$ with various $\lambda \in [0, 1]$.

The coefficients $\bar{\alpha}(\lambda)$ (see Table 2) computed numerically sharply coincide with those specified in (13) and (14) (with marginally faster for $\lambda = 0.9$).

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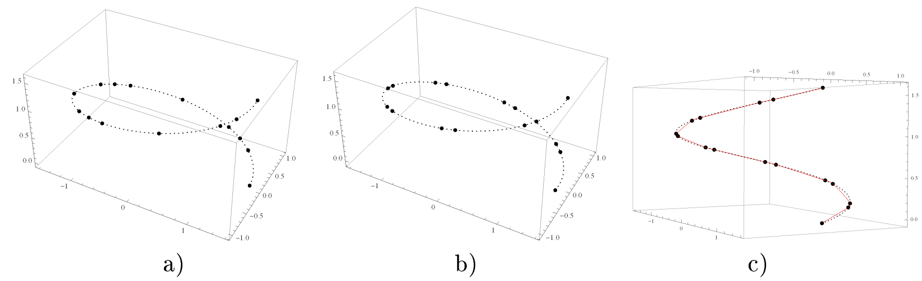

Fig. 2. *A quadratic helix* γ_{qh} (40) sampled along (dotted): a) (37) or b) (38) and c) fitted $\hat{\gamma}^{MC}$ with (38) & $\lambda = 0.5$ for $m = 15$.

Table 2. Computed $\bar{\alpha}(\lambda) \approx \alpha(\lambda)$ in (13) & (14) for γ_{qh} from (40) and various $\lambda \in [0, 1]$.

λ	0.0	0.1	0.3	0.5	0.7	0.9	1.0
$\bar{\alpha}(\lambda)$ for (37)	1.001	1.002	1.007	1.016	1.038	1.187	3.916
$\bar{\alpha}(\lambda)$ for (38)	0.001	1.001	0.005	1.017	1.056	1.322	3.908
$\alpha(\lambda)$ in Th. 2	1.0	1.0	1.0	1.0	1.0	1.0	4.0

Finally, *a conical spiral* γ_{cs} in $\mathbb{E}^3$ is tested.

Example 3. Let *a conical spiral* $\gamma_{cs} : [0, 1] \to \mathbb{E}^3$ be defined as follows:

$$\gamma_{cs}(t) = (2 \sin(0.5\pi t) \cos(2\pi t), 2 \sin(0.5\pi t) \sin(2\pi t), 2 \cos(0.5\pi t)). \qquad (41)$$

Fig. 3(a) (or Fig. 3(b)) contains the plots of γ_{cs} sampled more-or-less uniformly along either (37) or (38) (here $m = 15$).

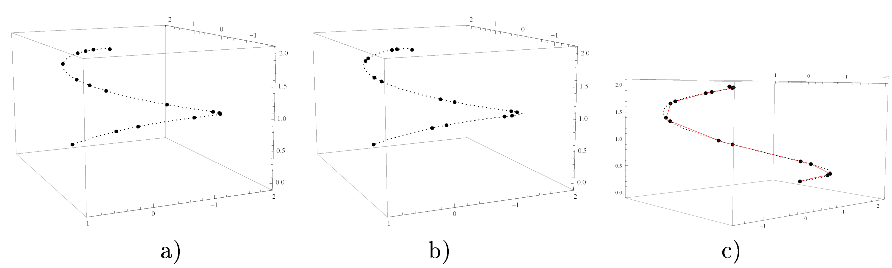

Fig. 3. *A conical spiral* γ_{cs} (41) sampled along (dotted): a) (37) or b) (38) and c) fitted $\hat{\gamma}^{MC}$ with (38) & $\lambda = 0.3$ for $m = 15$.

In order to compute $\bar{\alpha}(\lambda) \approx \alpha(\lambda)$ estimating the asymptotics from Th. 2 again a linear regression is used (as explained at the beginning of this section) for $60 \leq m \leq 120$ and varying $\lambda \in [0, 1]$. Table 3 enlists numerically computed estimates $\bar{\alpha}(\lambda) \approx \alpha(\lambda)$ for various $\lambda \in [0, 1]$ and for samplings (37) and (38).

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Evidently these numerical results re-emphasize the sharpness of the asymptotics determined by (13) and (14), with marginally faster case for $\lambda = 0.9$.

Table 3. Computed $\bar{\alpha}(\lambda) \approx \alpha(\lambda)$ in (13) & (14) for γ_{cs} from (41) and various $\lambda \in [0, 1]$.

λ	0.0	0.1	0.3	0.5	0.7	0.9	1.0
$\bar{\alpha}(\lambda)$ for (37)	0.999	1.002	1.008	1.019	1.051	1.264	3.939
$\bar{\alpha}(\lambda)$ for (38)	0.991	0.992	0.999	1.018	1.078	1.448	3.955
$\alpha(\lambda)$ in Th. 2	1.0	1.0	1.0	1.0	1.0	1.0	4.0

The experiments from this section confirm the asymptotics (and its sharpness) established in Th. 2 - see (13) and (14).

6 Conclusion

This work examines the asymptotics in approximating a regular parametric curve γ in $\mathbb{E}^n$ by a modified complete spline $\hat{\gamma}_3^{MC}$ (see Section 3) based on reduced data Q_m (sampled more-or-less uniformly (2)). The unknown interpolation knots $\mathcal{T}$ are compensated by $\hat{\mathcal{T}}$ with the aid of exponential parameterization (3) depending on a single parameter $\lambda \in [0, 1]$ and Q_m dispersion. The main theoretical contribution (see Th. 2) proves a linear convergence order in γ estimation by $\hat{\gamma}_3^{MC}$ for any $\lambda \in [0, 1)$. The numerical tests confirm *the sharpness* of both asymptotics from Th. 2 including the case of $\lambda = 1$, where a quartic convergence order in (14) prevails. Noticeably, though the case of $\lambda \in [0, 1)$ yields merely linear asymptotics (much slower than a quartic one for $\lambda = 1$) this case still provides one degree of freedom $\lambda \in [0, 1)$ to model the interpolant, should extra constraints on fitting Q_m are imposed. In particular, one may select the knots within the family (3) (i.e. with the optimal parameter $\lambda_{opt} \in [0, 1)$) to minimize the "acceleration mean" $\int_0^{\hat{T}} \|\hat{\gamma}''(\hat{t})\|^2 d\hat{t}$ (see e.g. [21], [22] and [23]). In contrast, such flexibility representing additional curve controlling tool is not available anymore for arbitrary fixed λ including the case of $\lambda = 1$. Such degree of freedom can still be preserved (once $\lambda \in [0, 1]$ is relaxed) at the cost of potentially decelerating the asymptotics (i.e. to a linear order) in trajectory estimation.

Related work and some applications (in *computer graphics and vision, image processing and engineering*) on fitting reduced data with various C^k (with $k = 0, 1, 2$) interpolation schemes $\hat{\gamma}$ based on alternative recipes $\hat{\mathcal{T}}$ to compensate the unknown knots $\mathcal{T}$ can be found e.g. in [1], [3], [8], [9], [10], [13], [16], [18], [19], [28], [30], [32], [34], [36], [37] or [40].

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