Simple error estimation for PIES in 2D elasticity problems

Agnieszka Bołtuć[0000-0002-4761-1170] and Eugeniusz Zieniuk[0000-0002-6395-5096]

University of Bialystok, Faculty of Computer Science, Bialystok, Poland
{a.boltuc,e.zieniuk}@uwb.edu.pl

Abstract. The paper presents a posteriori error estimation strategy in the parametric integral equation system (PIES). It uses collocation point differences between solutions obtained numerically by PIES and another solutions interpolated based on the initial PIES analysis. Various techniques for interpolation are proposed: by repeating the interpolation omitting one collocation point, by interpolating using only values from adjacent collocation points and by the one degree higher polynomial obtained using the least squares approximation. This allows the calculation of local and global percentage error using the integral over the mentioned differences. Finally, it can be applied to ensure convergence of the solutions using the PIES method or to adaptive refinement of distribution or number of collocation points by identifying boundary regions where the error is relatively high.

Keywords: Error estimation, Parametric integral equation system (PIES), Collocation points, Distribution refinement.

1 Introduction

Parametric integral equation system (PIES) is the method for solving boundary value problems, developed by the authors as an alternative to well-known numerical approaches like the finite element method (FEM) [1,2,3], the boundary element method (BEM) [4,5,6] or so-called meshless methods [7,8,9]. A distinctive aspect of PIES lies in its approach to the geometry representation. Instead of relying on meshing or discretization (into various kinds of elements), the method uses a limited set of points: corner points of the polygonal geometry or key boundary points reflecting the curvilinear shape of the considered body. It is possible, because PIES is an analytical modification of the boundary integral equation (BIE) [4,5,6], consisting of the analytical incorporation of the shape into the formalism of the equation. For this reason, the shape can be represented in various ways, but the authors chose very effective and flexible parametric curves (e.g. Bezier), which are well-known in computer graphics [10,11]. They allow for simple modeling using dedicated control points. The analytically incorporated shape, described using formulas representing curves, means that each shape modification is automatically reflected in the PIES formalism.

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Moreover, the proposed approach enables a clear separation between shape modeling and solution approximation - two fundamental stages in solving boundary value problems. In FEM and BEM, they are often dependent on each other, which means more elements equal more accurate solutions. The PIES solution is approximated by a series with arbitrary basis functions (e.g. Lagrange polynomials in this paper) and forces the PIES equation to be satisfied at selected points (collocation points). Such an approach allows for the accuracy of the solutions to be influenced by changing only one parameter of the approximating series without the need to perform cumbersome re-discretization, as is the case with the methods mentioned above. Meshless methods also do not rely on traditional elements like PIES. However, despite their variations - such as the boundary node method (BNM) [12], where input data includes only boundary nodes - they still require domain partitioning (division into cells) for integration.

PIES has been successfully applied to solving various boundary value problems, starting with potential problems [13], through acoustic [14], elastic [15], and transient heat conduction [16] to elastoplastic [17]. The effectiveness of the described above way of shape modeling was then examined, and the accuracy of the obtained solutions was compared to the analytical results. The conclusions of this analysis were very satisfactory. However, an analytical solution only sometimes exists, and there may also be situations where there is no even numerical solution to compare. Therefore, developing a dedicated error estimation strategy for PIES is essential. It refers to a set of techniques designed to evaluate the accuracy of numerical solutions to boundary value problems without comparing it to any other exact results. They are crucial to reliably assess solution accuracy, guarantee convergence, and enable adaptive refinement, ensuring the method's robustness and practical applicability.

The error estimation schemes may differ between methods for solving boundary value problems [18,19,20]. Focusing on BEM, as it is a predecessor of PIES, they can be classified into several types: the residual type, the interpolation type, the integral equation type, the node sensitivity type and the solution difference type [21]. Since PIES aims to improve efficiency (by e.g. simplifying modeling), from available approaches was selected the one which is computationally cheaper than the others – interpolation error estimation. It has been widely used in other numerical methods [19,20], even though the accuracy of the predicted solutions is not guaranteed. This approach generally compares the original numerical solution obtained by the applied method with the interpolated solution (from now on referred to as the predicted solution). The interpolation is made based on the initial numerical solution mentioned above. The difference between predicted and numerical solutions is estimated as the error.

This paper presents three approaches to interpolation of the predicted solution in PIES. The first consists of omitting one of the collocation points (the one where a new solution is calculated) and re-interpolating by the Lagrange polynomial based on the existing PIES solutions in the remaining collocation nodes. In the second variant, the Lagrange interpolation is performed only based on the values from the neighboring collocation nodes (one from the left and two from the right of the estimated collocation point). The last approach uses the least square approximation with a polynomial of arbitrary degree based on the PIES values in the collocation points, skipping the currently

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predicted one. A sequence of monomials is assumed as the basis function. The relative percentage errors, local and global, are calculated as integrals over the boundary from the obtained differences between the initial numerical solution and the predicted solution. Some examples have been used to demonstrate the behavior of the error estimators.

2 Parametric integral equation system (PIES) for 2D elasticity

A parametric integral equation system (PIES) for 2D elastic problems without body forces can be presented in the following form [22]

$$0.5\boldsymbol{u}_l(\bar{s}) = \sum_{j=1}^{n}\int_{s_{j-1}}^{s_j}\{\overline{\boldsymbol{U}}_{lj}^{*}(\bar{s},s)\boldsymbol{p}_j(s)-\overline{\boldsymbol{P}}_{lj}^{*}(\bar{s},s)\boldsymbol{u}_j(s)\}J_j(s)ds, \tag{1}$$

where $s_{l-1} \le \bar{s} \le s_l$, $s_{j-1} \le s \le s_j$, $l = 1,2,3,\dots,n$, $J_j(s)$ is the Jacobian of transformation to a 1D parametric reference system in which the boundary in PIES is defined and n is the number of boundary segments.

The first integrand, $\overline{\boldsymbol{U}}_{lj}^{*}(\bar{s},s)$, is a modified fundamental solution and, as mentioned in the introduction, takes into account in its mathematical formalism the shape of the boundary defined in a general way. For the plane strain state, it can be represented by [22]

$$\overline{\boldsymbol{U}}_{lj}^{*}(\bar{s},s) = -\frac{1}{8\pi(1-v)\mu}\begin{bmatrix}(3-4v)\ln(\eta)-\frac{\eta_1^2}{\eta^2} & -\frac{\eta_1\eta_2}{\eta^2}\\ -\frac{\eta_1\eta_2}{\eta^2} & (3-4v)\ln(\eta)-\frac{\eta_2^2}{\eta^2}\end{bmatrix}, \tag{2}$$

where $\eta = [\eta_1^2+\eta_2^2]^{0.5}$, $\eta_1 = \Gamma_j^{(1)}(s)-\Gamma_l^{(1)}(\bar{s})$, $\eta_2 = \Gamma_j^{(2)}(s)-\Gamma_l^{(2)}(\bar{s})$, and v is Poisson's ratio and μ is a shear modulus. The parametric function $\boldsymbol{\Gamma}(s)$ can be represented by various curves known from computer graphics [10,11].

The following expression $\overline{\boldsymbol{P}}_{lj}^{*}(\bar{s},s)$ represents the second kernel [22]

$$\overline{\boldsymbol{P}}_{lj}^{*}(\bar{s},s) = -\frac{1}{4\pi(1-v)\eta}\begin{bmatrix}P_{11} & P_{12}\\ P_{21} & P_{22}\end{bmatrix}, l,j = 1,2,\dots n, \tag{3}$$

where

$$P_{11} = \left\{(1-2v)+2\frac{\eta_1^2}{\eta^2}\right\}\frac{\partial\eta}{\partial n}, P_{22} = \left\{(1-2v)+2\frac{\eta_2^2}{\eta^2}\right\}\frac{\partial\eta}{\partial n},$$

$$P_{21} = P_{12} = \left\{2\frac{\eta_1\eta_2}{\eta^2}\frac{\partial\eta}{\partial n}-(1-2v)\left[\frac{\eta_1}{\eta}n_2(s)+\frac{\eta_2}{\eta}n_1(s)\right]\right\},$$

$$\frac{\partial\eta}{\partial n} = \frac{\partial\eta_1}{\partial\eta}n_1(s)+\frac{\partial\eta_2}{\partial\eta}n_2(s),$$

and $n_1(s)$ and $n_2(s)$ are direction cosines of the external normal to jth segment of the boundary.

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Functions $\boldsymbol{u}_j(s)$ and $\boldsymbol{p}_j(s)$ are parametric boundary functions. They are known or searched depending on individual segments' boundary conditions. They can be approximated using series with arbitrary basis functions

$$\boldsymbol{u}_j(s) = \sum_{k=0}^{M-1} \boldsymbol{u}_j^{(k)} L_j^{(k)}(s), \quad \boldsymbol{p}_j(s) = \sum_{k=0}^{M-1} \boldsymbol{p}_j^{(k)} L_j^{(k)}(s), j = 1, \ldots, n, \tag{4}$$

where $\boldsymbol{u}_j^{(k)}, \boldsymbol{p}_j^{(k)}$ are unknown coefficients, M is the number of coefficients on segment j and $L_j^{(k)}(s)$ is the Lagrange polynomial on segment j. Various polynomials can be used as basis functions, like Chebyshev, Legendre, etc., but the Lagrange polynomials are applied in this paper.

Equation (1) is then written for all collocation points and takes the form of the equation system with $2 \times n \times M$ equations. After solving it, the unknown coefficients from (4) are obtained. In the case of Lagrange polynomials, they are solutions at collocation points at the same time.

The accuracy of PIES solutions depends on two main factors: the number M in (4) and the arrangement of collocation points on each boundary segment. Various approaches have been tested over time, but two are mostly used: uniform distribution and at places corresponding to the roots of Chebyshev polynomials of the first kind (degree M). Since the recursive formula by which the roots of the Chebyshev polynomial are generated is known [23], the arrangement of collocation points at the locations of these roots is automatic. The examination of the convergence technically comes down to the choice of just two parameters: the number M and the way of arranging the collocation points.

3 Boundary modeling by curves

The main advantage of PIES is that the approximation of the solutions and the shape are separated. It comes from the fact that the boundary is modeled using parametric curves known from computer graphics [10,11]. Instead of classical discretization used in BEM or FEM, the whole segments are created by a single curve. Moreover, they are incorporated into the PIES formula analytically, so each change in the shape of the curve involves an automatic modification of the PIES formalism. The curves are easily modified by changing only their control points. This process is incomparably more effective than re-discretization in the so-called element methods. Fig. 1 presents how the boundary is modeled depending on whether the segment is straight or curved.

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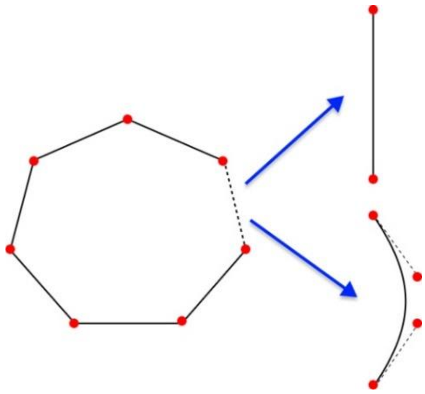

Fig. 1. Modeling of the boundary in PIES.

As shown in Fig. 1, the boundary is modeled using curves. This paper uses Bezier curves of various degrees depending on the required shape. Thus, the polygonal shapes are modeled by curves of the first degree, defined by only two control points, while curved shapes by the curves of the third degree using four control points. Higher degrees are unnecessary because cubic curves can model all the necessary shapes and are not too computationally expensive.

Modification of such defined geometry is also straightforward. It is enough to change the position of small number of control points to modify the shape significantly. An example is shown in Fig. 2, where three control points of the curved geometry change its shape.

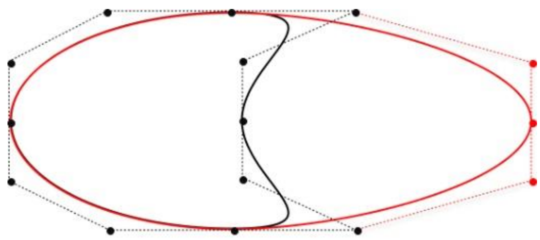

Fig. 2. Modification of the boundary in PIES.

The above-described incorporation of curves into the PIES formalism allowed for separating the shape approximation from the solution approximation, which in PIES bases on changing the parameters of the approximating series (4).

4 Error estimation

The exact error of the boundary solutions is calculated as

$$\boldsymbol{e}_u = \boldsymbol{u} - \boldsymbol{u}_n, \quad \boldsymbol{e}_p = \boldsymbol{p} - \boldsymbol{p}_n, \tag{5}$$

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where $\boldsymbol{u}, \boldsymbol{p}$ are the exact solutions and $\boldsymbol{u}_n, \boldsymbol{p}_n$ are the numerical displacements and tractions obtained by PIES. It is known that analytical solutions only exist for some engineering problems. Therefore, its prediction should be used instead. For this reason, formula (5) takes the following form

$$\check{\boldsymbol{e}}_u = \check{\boldsymbol{u}} - \boldsymbol{u}_n, \quad \check{\boldsymbol{e}}_p = \check{\boldsymbol{p}} - \boldsymbol{p}_n, \tag{6}$$

where $\check{\boldsymbol{u}}, \check{\boldsymbol{p}}$ are the predicted approximations of displacements and tractions.

The predicted approximations $\check{\boldsymbol{u}}, \check{\boldsymbol{p}}$ can be obtained by interpolating the PIES original numerical solution using various-degree polynomials. The original PIES solution is just a solution of the system of equations at all collocation points. Therefore, the first approach (in the following chapters called approach 1) determines the new prediction at the particular collocation point by re-interpolating based on the original values, omitting the considered node. As shown in Fig. 3, if the new value is predicted for node 2, the interpolation is performed using the remaining nodes (without 2). The same interpolation approach (Lagrange polynomials) is applied as in the PIES method.

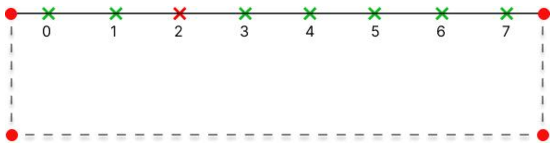

Fig. 3. The boundary segment with collocation nodes - diagram for approach 1.

The second approach (approach 2) concerns interpolation using only the nearest neighbors. To have the second-degree polynomial, three neighbors are considered - one to the left of the considered node and two to the right. For example, in Fig. 4, to predict the value of node 2, values from nodes 1,3,4 should be interpolated. There are some extreme cases: for the right-most node and the penultimate node, where an appropriate number of nodes from the left are taken into account to preserve the degree of the polynomial. The left-most node takes into account only nodes from its right.

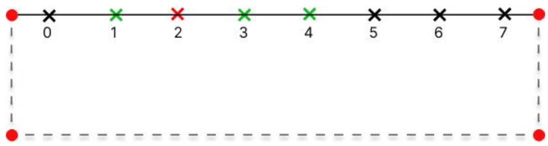

Fig. 4. The boundary segment with collocation nodes - diagram for approach 2.

Both presented above ideas consider the interpolation of the numerical solution by a lower degree of interpolation function than in the initial PIES analysis. Therefore, the third approach (approach 3) is also proposed. It uses least squares approximation [23]. To obtain the interpolation polynomial

$$Q_m(x) = a_0 + a_1 x + \cdots + a_m x^m, \tag{7}$$

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one must choose its degree (m) and then find the coefficients ($a_0, a_1, \dots, a_m$). They are chosen so that the value of the squared deviation between the polynomial values ($Q_m(x_i)$) on the set of collocation points and the PIES initial results ($f(x_i)$) is as small as possible

$$S = \sum_{i=1}^{n}[Q_m(x_i) - f(x_i)]. \tag{8}$$

By finding partial derivatives (8) concerning all coefficients of the polynomial (7) and equating them to zero, a system of m+1 equations is obtained, based on which the required coefficients of the polynomial (7) are determined. Then (7) can be used to predict new values at collocation points. This time, like in the first approach, values from all collocation nodes are used except the considered one (Fig. 3), and the polynomial degree for predicting is one bigger than in the initial PIES analysis.

Errors (6) are estimated collocation points errors. To calculate the error for the boundary segment i, the following equations can be used

$$\|\boldsymbol{e}_u\|_i = \int_{\Gamma_i} (\breve{\boldsymbol{u}} - \boldsymbol{u}_n) d\Gamma\,, \ \|\boldsymbol{e}_p\|_i = \int_{\Gamma_i} (\breve{\boldsymbol{p}} - \boldsymbol{p}_n) d\Gamma. \tag{9}$$

The global error for the whole boundary is expressed by

$$\|\boldsymbol{e}_u\| = \int_{\Gamma} (\breve{\boldsymbol{u}} - \boldsymbol{u}_n) d\Gamma\,, \ \|\boldsymbol{e}_p\| = \int_{\Gamma} (\breve{\boldsymbol{p}} - \boldsymbol{p}_n) d\Gamma. \tag{10}$$

The global relative percentage error $\boldsymbol{\eta}$ is then written as

$$\boldsymbol{\eta}_{\boldsymbol{u}} = \frac{\|\boldsymbol{e}_u\|}{\|\boldsymbol{u}_n\|} \times 100\%, \boldsymbol{\eta}_{\boldsymbol{p}} = \frac{\|\boldsymbol{e}_p\|}{\|\boldsymbol{p}_n\|} \times 100\%. \tag{11}$$

5 Numerical examples

5.1 Initial verification of proposed approaches for global error estimation in comparison to analytical solutions

The first example concerns a cylinder subjected to an internal pressure $p = 22.5MPa$ under plane strain conditions. Young's modulus $E = 21000MPa$ and Poisson's ratio $v = 0.3$ are the elastic material properties. Fig. 5 presents the quarter of the cylinder because of the symmetry.

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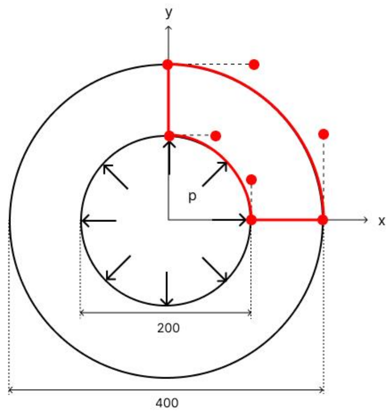

Fig. 5. The quarter of the cylinder under internal pressure.

As can be seen, the geometry is modeled using four curves: two of the first degree and two of the third degree. Only eight nodes are defined (●).

As mentioned earlier, the accuracy of solutions depends on the arrangement and the number of collocation points. The two most tested variants are: uniform and at places corresponding to the roots of Chebyshev polynomials. As the interpolation is performed on collocation nodes, the uniform variant is excluded from the tests, because at high degrees of the interpolation polynomial we can expect distortions known as Runge's phenomenon [24].

Global relative percentage errors for the whole boundary (11) for solutions u_1, p_1 are calculated. Fig. 6 presents values obtained for 4-20 collocation nodes at each boundary segment, and three approaches are used to predict new values at existing nodes (section 4).

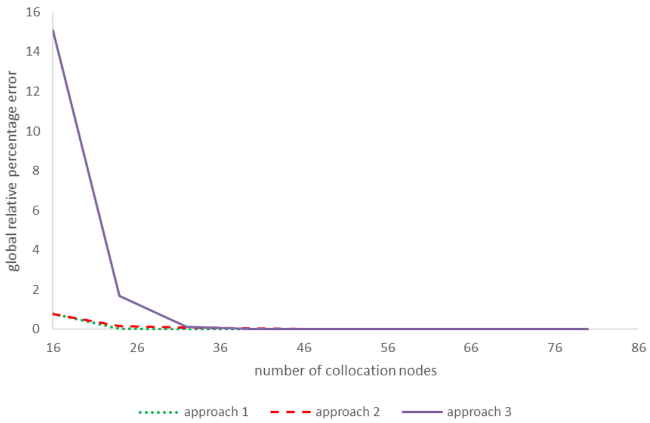

Fig. 6. Global relative percentage error η_{u_1}.

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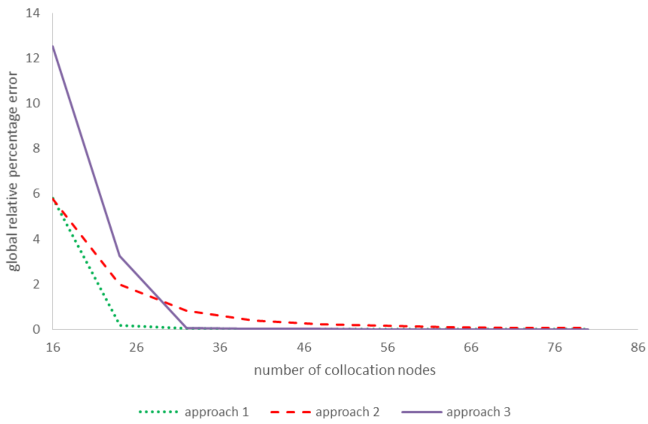

Fig. 7. Global relative percentage error η_{p_1}.

As shown in Fig. 6 and 7, global relative percentage errors calculated by approaches 1,2 and 3 have similar trends for both u_1 and p_1. Starting from higher values, they stabilize when using 8-10 collocation points per segment (32-40 in total). With 20 (80 in total) collocation points used, of the three considered estimation approaches, the second method (interpolation using only nearest neighbors, always of degree 2) had the most significant error. The other two are characterized by the similar level of error.

The analytical results [25] confirm the trend in Fig. 6 and 7. Displacements u_1 were calculated at the lower linear boundary of the quarter. The average relative error decreased from 0.1035% (16 collocation nodes in total) to 0.0277% (80 collocation nodes in total), but as shown in Fig. 6 the almost final stable level is reached at 32 collocation points (0.0278%).

5.2 Proposed approaches in applications for global and local error estimating

In the second example, the elastic plate with the circular hole subjected to the tensile load p at its ends is considered (Fig. 8). Plane stress conditions are assumed with the following material properties $E = 1 \times 10^6 MPa$ and $v = 0.3$. Due to the symmetry, only the upper right square quadrant is analyzed.

The boundary is modeled in PIES using five curves: four of the first and one of the third degrees. It requires posing seven nodes.

Once again, the error estimation is performed using various numbers of collocation points and multiple approaches for solution interpolation.

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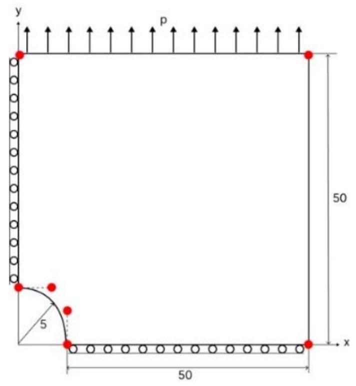

Fig. 8. The quarter of the plate with the circular hole.

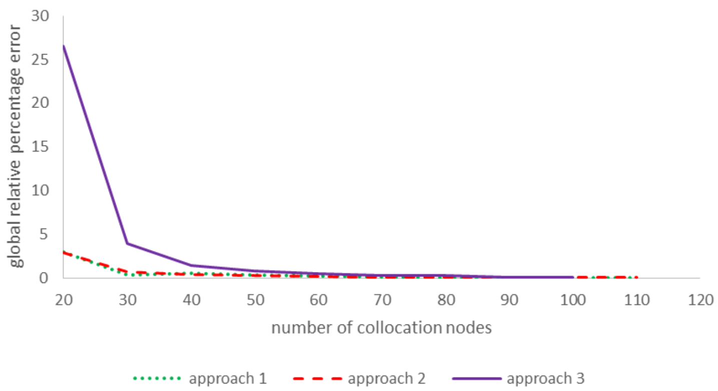

Fig. 9. Global relative percentage error η_{u_1}.

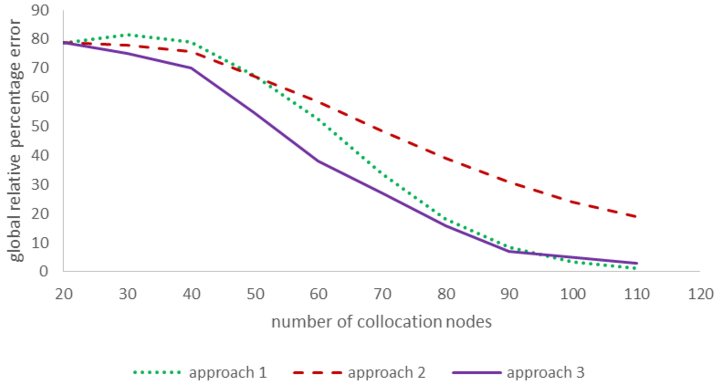

Fig. 10. Global relative percentage error η_{p_1}.

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As shown in Fig. 9, error η_{u_1} for all approaches has the same trend and similar values. Once again, stabilization is visible with about 50 colocation points, where all error values are under 1%. The most significant value of η_{u_1} for 110 points is obtained by approach 2, which confirms the results from the previous example.

Analyzing η_{p_1} (Fig. 10) shows that all approaches start with a similar high error (~79%). With the highest number of collocation points, approaches 1 and 3 have their error reduced to 1-3%, while approach 2 remained at a high level of 19% error. It is still the approach with the most significant estimated error value.

Finally, analytical results [25] are also analyzed. Solution p_2 at the bottom linear boundary of the plate is calculated. The comparison of the exact with the numerical results obtained by PIES with 20 and 110 collocation nodes is presented in Fig. 11. As can be seen, there is a very significant improvement in the accuracy of the solutions with the increased number of collocation nodes. It should be emphasized that the numerical solutions for 70 collocation points already have an error close to that achieved with 110 points.

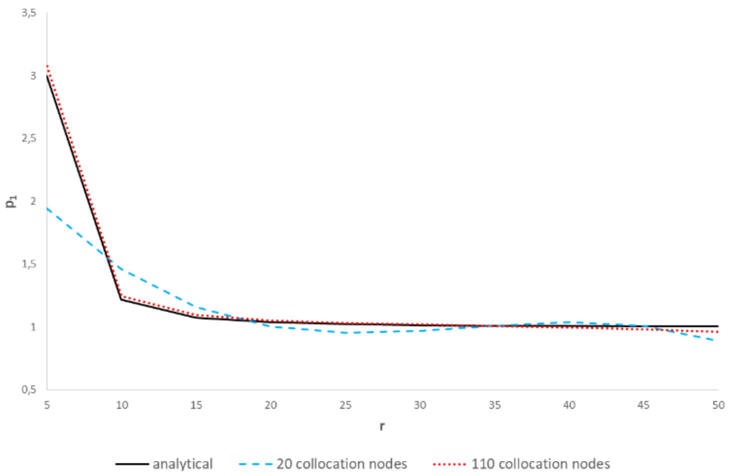

Fig. 11. Comparison of analytical and PIES results for various number of collocation nodes.

Analyzing errors more locally, i.e., on individual segments, it can be seen that the most significant relative percentage error was obtained for p_2 on the left vertical side and p_1 on the bottom side of the plate. It amounted to about 3% and 1% respectively. The highest error at the collocation point on the vertical segment is at the first point from the hole. The difference expressed by formula (6) is 0.012906. At the bottom side it is again the closest collocation node to the hole, with a difference equal to 0.0468041. The calculations described in this paragraph were performed using approach 3. Having the sides and points with the highest errors, it is possible to densify the collocation points to improve the accuracy of the obtained results.

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5.3 Global and local error estimating on example with numerical solutions only

The last example concerns the polygonal L-shaped plate under uniform tension and plane stress conditions. The boundary conditions and dimensions are presented in Fig. 12. The assumed material properties are $E = 1 \times 10^5 \, {}^{N}/_{cm^2}$ and $\nu = 0.3$.

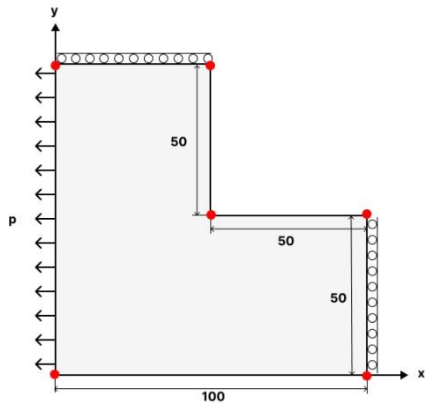

Fig. 12. The L-shaped plate under the uniform tension.

The considered polygonal boundary is modeled using six curves of the first degree by six corner points.

Global relative percentage errors for u_1 and p_1 are calculated by taking various numbers of collocation nodes and applying multiple approaches proposed in the paper. The results are presented in Tables 1 and 2.

Table 1. Global relative percentage error η_{u_1}.

no of collocation points	η_{u_1}		
	approach 1	approach 2	approach 3
24	4.4496	4.4496	27.304
36	0.8971	1.0572	3.9844
48	0.2671	0.5383	0.3969
60	0.1201	0.3124	0.6612
72	0.0718	0.1955	0.1376
84	0.0481	0.1301	0.1687
96	0.0353	0.0902	0.0915
108	0.0269	0.066	0.0433
120	0.0211	0.0496	0.0469

As can be seen from Tables 1 and 2, each approach generates a similar trend. For both η_{u_1} and η_{p_1}, it can be observed that since using 48 points, estimated errors are less than

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1% for all proposed approaches Values obtained at the maximum number of collocation points are close to zero and similar in each tested approach.

Table 2. Global relative percentage error η_{p_1}.

no of collocation points	η_{p_1}		
	approach 1	approach 2	approach 3
24	2.5624	2.5624	38.2614
36	0.7162	0.2973	7.9506
48	0.0486	0.1059	0.1371
60	0.0768	0.0546	0.1236
72	0.0297	0.0313	0.0188
84	0.0262	0.0201	0.0219
96	0.0194	0.0139	0.0197
108	0.0156	0.0101	0.0104
120	0.0125	0.0074	0.0093

These conclusions are confirmed by analyzing numerical solutions obtained for different numbers of collocation points. Such a test was carried out due to the lack of analytical solutions for this example. Tractions p_1 at the right boundary of the L-shaped plate are compared for 24, 48, 120, and 180 collocation points and presented in Table 3.

Table 3. Values of p_1 at the right edge of the plate.

y	no of collocation points			
	24	48	120	180
5	13.5781	13.1634	13.2307	13.2247
10	58.4243	60.4602	60.4068	60.4028
15	104.528	107.376	107.288	107.285
20	151.543	153.99	153.977	153.976
25	199.125	200.539	200.575	200.576
30	246.927	247.102	247.118	247.119
35	294.603	293.545	293.541	293.544
40	341.809	339.659	339.719	339.726
45	388.198	385.421	385.549	385.558

As can be seen, solutions obtained for 48 collocation nodes are very similar to those for 120 and, finally, 180 points. Therefore, if we assume that the solution for the most significant number of collocation points is exact, the obtained global relative percentage errors reflect the results shown in Table 3.

The boundary relative percentage errors are calculated to use the proposed estimation approach to solutions' adaptive refinement. The approach 3 with 96 collocation points is used. The most significant errors were obtained for the sides at the concave corner and were 0.1783% (u_1) and 0.3086% (u_2) for the horizontal and vertical edges, respectively. This may mean that the density of collocation points on these sides improves the final accuracy. Such a process can be done automatically.

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6 Conclusions

The paper presents the approach to error estimating in PIES for 2D elastic problems using various techniques for obtaining predicted solutions. The first performs interpolation at all collocation nodes with received numerical solutions besides the one where the expected value is searched. The second one carries out interpolation more locally, based on three neighboring collocation nodes. The last one uses the approximation polynomial of a degree one greater than the number of collocation points (without the node at which the value is predicted) obtained by the least square approximation.

The proposed approaches are applied to three elasticity examples. Both the global and local (boundary) relative percentage errors are calculated for different numbers of collocation points using three techniques to obtain predicted solutions. In general, all proposed approaches gave similar error trends. Approach 3 always starts with the most significant error value, while approach 2 usually ends with it. The errors decreased with the increasing number of collocation points, so in most cases they fell to 1-3% with a total number of points of 80-120, depending on the example. Moreover, the local analysis is also reliable because the analyzed approaches returned the most significant errors on those boundaries that are actually characterized by the most considerable variability of solutions and thus require an increase in the number and change in the arrangement of collocation points.

The next step in research carried out by the authors should be the implementation of adaptive refinement of the number and distribution of collocation points based on the error estimation in PIES. Combined with optimization algorithms, it can be an excellent tool to improve the accuracy of the obtained PIES solutions.

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