Using B-spline function properties in the PIES method to handle singularities in boundary value problems

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Abstract. The paper presents the formulation of a parametric integral equation system (PIES) for 2D boundary value problems with singular solutions. The proposed approach combines B-spline basis functions with the PIES formalism to approximate the results in challenging problems such as notched plates or concentrated loads. The B-spline function with a specific degree and knots is used in approximating series instead of previously applied polynomials (such as Lagrange or Chebyshev). Increasing knot multiplicity to reduce continuity at some locations or increasing the number of knots in singular regions allows for more accurate results. The proposed approach is validated through selected problems. The results confirm that the B-spline-enhanced PIES approach can effectively solve singular problems with improved accuracy.

Keywords: Parametric integral equation system (PIES) · B-spline basis functions · Knots · Singular solutions.

1 Introduction

The parametric integral equation system (PIES) [1–4] is one of the methods that is used to solve boundary value problems. The authors of the paper develop it as an alternative to known numerical methods such as the finite element method (FEM) [5–7] and the boundary element method (BEM) [7–9]. The main difference between PIES and the above methods is that PIES does not require classical discretization. It uses parametric curves [10, 11] to model the boundary. Each segment of the boundary is defined globally by a single curve. The boundary segments are created naturally on the basis of the shape. Polygonal shapes have segments that coincide with each side, whereas in the case of curved shapes, the way they are divided depends on their complexity. In both cases, modeling is limited to assigning a small number of control points. Such defined shapes are analytically incorporated into the mathematical formalism of PIES. Therefore, each modification of the shape results in a modification of the integral equation. The authors have used mainly the Bezier and NURBS curves for this purpose.

Using the appropriate number of curves of the appropriate degree ensures the accuracy of shape modeling. The accuracy of the solutions is independently

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controlled and determined by entirely different factors. PIES solutions are approximated using series with a specific number of terms and with selected basis functions. Changing both of these parameters affects the accuracy. So far, the authors have analyzed various numbers of series terms and various polynomials (such as Chebyshev, Legendre, Lagrange) as basis functions [1–4]. However, when solutions are singular, which is common in boundary value problems, such an approach may give unsatisfactory results. Using a high-degree polynomial in such situations is necessary, which may lead to numerical instability (Runge's phenomenon). Moreover, these polynomials are global over the range of approximation, so they cannot accurately take the local singularity into account. It would be appropriate to consider functions for which accuracy can be improved only in the singularity regions without affecting the entire segment. Such functions are B-spline, described by degree and knots. The use of these functions for the interpolation or approximation of solutions instead of polynomials is known [12]. Their properties allow for practical application in complex functions with high fluctuations. Therefore, it was decided to use them, among other basis functions, in the efficient PIES for demanding singularity problems.

B-spline functions are also used in other methods to solve boundary value problems, such as isogeometric FEM and BEM [13, 14] or meshless methods [15]. However, the idea behind these methods differs from that of PIES. The lack of elements is a common feature of meshless methods and PIES. However, the way geometry is represented is entirely different. Meshless methods use scattered nodes in the domain. Lack of structured connectivity can cause higher computational costs and problems with boundary condition implementation. PIES uses a parametric boundary representation. Both approaches can apply B-spline functions to approximate solutions. On the other hand, isogeometric methods are known for using B-spline functions for both modeling and approximation of solutions. In PIES, it is also possible, but these two stages are separated, and various methods can be applied independently. Moreover, all the abovementioned approaches (besides PIES) still divide the domain/boundary into some cells/elements to integrate numerically.

This paper proposes the use of B-spline functions to approximate solutions in PIES. This allows for more accurate results for some singular cases, still without changing the way of modeling. The shape is defined using Bezier curves to emphasize the independence of shape approximation from solution approximation. Some examples are considered. They are solved using PIES with the proposed B-spline functions and previously applied Chebyshev polynomials. It is also investigated how B-spline properties (degree and knots) changed by, e.g., knot multiplicity, clamped knots or adaptive knot placement can affect the accuracy of solutions. Two examples analyzed confirm the accuracy of the proposed approach.

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2 Parametric integral equation system (PIES)

The parametric integral equation system (PIES) is presented in the example of 2D elastic problems without body forces [3]

$$0.5\boldsymbol{u}_l(\bar{s}) = \sum_{j=1}^{n} \int_{s_{j-1}}^{s_j} \left\{ \mathbf{U}^*_{lj}(\bar{s},s)\boldsymbol{p_j}(s) - \mathbf{P}^*_{lj}(\bar{s},s)\boldsymbol{u_j}(s) \right\} J_j(s)ds, \tag{1}$$

where $l,j = 1,2,...,n$, $s_{l-1} \leq \bar{s} \leq s_l$, $s_{j-1} \leq s \leq s_j$. $J_j(s)$ is a Jacobian of transformation to the parametric reference system and n is the number of segments in the considered boundary. The parametric boundary functions $\boldsymbol{p_j(s)}$ and $\boldsymbol{u_j(s)}$ are known or searched on the particular segments of the boundary.

The boundary fundamental solution $\mathbf{U}^*_{lj}(\bar{s},s)$ from (1) is presented in the following form [3]

$$\mathbf{U}^*_{lj}(\bar{s},s) = -\tfrac{1}{8\pi(1-\nu)\mu} \begin{bmatrix} (3-4\nu)ln(\eta) - \frac{\eta_1^2}{\eta^2} & -\frac{\eta_1\eta_2}{\eta^2} \\ -\frac{\eta_1\eta_2}{\eta^2} & (3-4\nu)ln(\eta) - \frac{\eta_2^2}{\eta^2} \end{bmatrix}, \tag{2}$$

where $\eta_1 = \mathit{\Gamma}_j^{(1)}(s) - \mathit{\Gamma}_l^{(1)}(\bar{s})$, $\eta_2 = \mathit{\Gamma}_j^{(2)}(s) - \mathit{\Gamma}_l^{(2)}(\bar{s})$, $\eta = \left[\eta_1^2 + \eta_2^2\right]^{0.5}$, ν is Poisson's ratio and μ is the shear modulus.

The second integrand $\mathbf{P}^*_{lj}(\bar{s},s)$ from (1) is also presented in the matrix form by [3]

$$\mathbf{P}^*_{lj}(\bar{s},s) = -\frac{1}{4\pi(1-\nu)\eta} \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}, \tag{3}$$

where

$$P_{11} = \left\{(1-2\nu) + 2\frac{\eta_1^2}{\eta^2}\right\}\frac{\partial\eta}{\partial n}, \quad P_{22} = \left\{(1-2\nu) + 2\frac{\eta_2^2}{\eta^2}\right\}\frac{\partial\eta}{\partial n},$$

$$P_{21} = P_{12} = \left\{2\frac{\eta_1\eta_2}{\eta^2}\frac{\partial\eta}{\partial n} - (1-2\nu)\left[\frac{\eta_1}{\eta}n_2(s) + \frac{\eta_2}{\eta}n_1(s)\right]\right\},$$

and $\frac{\partial\eta}{\partial n} = \frac{\partial\eta_1}{\partial n}n_1(s) + \frac{\partial\eta_2}{\partial n}n_2(s)$, while $n_1(s)$ and $n_2(s)$ are direction cosines of the external normal to jth segment of the boundary.

The shape of the boundary is directly defined in the kernels (2) and (3) by parametric curves $\boldsymbol{\mathit{\Gamma}}$ of arbitrary degree [10, 11]. The following section shows that they can efficiently model and modify various boundary shapes.

The form of PIES for other types of problems like potential, acoustic or elastoplastic can be found, among others, in [1, 2, 4].

3 Modeling of 2D geometry in PIES

As mentioned in the introduction, the boundary in PIES is modeled entirely by curves. The most popular, among others, are the Bezier, B-spline, or NURBS

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curves [10, 11]. These curves can be polynomials of any degree. The most commonly used polynomials are first-degree to model polygonal domains and third-degree to model curved domains. The main advantage of parametric curves is that their definition requires a small number of control points. In polygonal problems, only corner points can be posed.

Parametric curves are analytically incorporated into the PIES formalism (1). As a result, the PIES is defined on a straight line in a parametric reference system for an arbitrary shape. The shape is modeled and modified by the control points mentioned above. This process is independent of approximating the boundary functions (solutions). For this reason, in this paper, Bezier curves were used to model the shape of the boundary instead of NURBS to emphasize the independence of the shape approximation from the approximation of the solutions. The use of NURBS curves for some shapes would probably additionally improve the accuracy of the solutions. An example of modeling the polygonal and curved domain in PIES using Bezier curves is shown in Fig. 1.

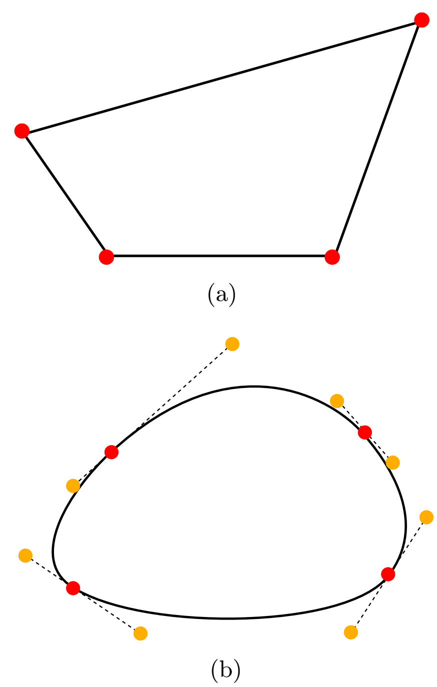

Fig. 1: The way of modeling in PIES: a) polygonal shape, b) curved shape.

Both shapes presented in Fig. 1 consist of four segments. The polygonal shape (Fig. 1a) is built from linear segments defined by first-degree Bezier curves, while the curved one (Fig. 1b) is defined by third-degree curves. Both are modeled using only control points.

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4 Approximation of PIES

The PIES solution is reduced to finding the unknown functions $\boldsymbol{u_j(s)}$ and $\boldsymbol{p_j(s)}$ from (1) on individual boundary segments. They are approximated by the following series [3]

$$\boldsymbol{u}_j(s) = \sum_{k=0}^{M-1} \boldsymbol{u}_j^{(k)} f_j^{(k)}(s), \quad \boldsymbol{p}_j(s) = \sum_{k=0}^{M-1} \boldsymbol{p}_j^{(k)} f_j^{(k)}(s), \tag{4}$$

where $\boldsymbol{u}_j^{(k)}$, $\boldsymbol{p}_j^{(k)}$ are searched coefficients, M is the number of coefficients on segment j, $f_j^{(k)}(s)$ are arbitrary basis functions and $j = 1, 2, ..., n$.

So far, Legendre, Chebyshev or Lagrange polynomials have been used as basis functions [3, 4]. However, modeling singular solutions (e.g. near cracks, notches, corners) requires functions that effectively deal with local discontinuities and rapidly changing values. The polynomials mentioned above are global, so it is impossible to consider the local singularity. An increase in their degree can help, but also lead to numerical instability. Therefore, the authors decided to use B-spline basis functions that provide more local control, allowing for a more accurate capture of strong gradients near singularities.

A B-spline basis function of degree q is defined recursively using the Cox-de Boor recurrence formula starting with a piecewise constant polynomial ($q = 0$) [16]

$$N_{i,0}(v) = \begin{cases} 1 & if \quad v_i \leq v < v_{i+1}, \\ 0 & otherwise, \end{cases} \tag{5}$$

where v_i are knots. They are sorted in non-decreasing order in vector $\boldsymbol{V}$, and their role is partitioning the parameter space. The number of knots is $M + q + 1$.

For $q = 1, 2, 3, ...$ B-splines are defined by [16]

$$N_{i,q}(v) = \frac{v - v_i}{v_{i+q} - v_i} N_{i,q-1}(v) + \frac{v_{i+q+1} - v}{v_{i+q+1} - v_{i+1}} N_{i+1,q-1}(v). \tag{6}$$

B-spline functions have properties that affect the accuracy of modeling singular solutions, such as their degree and the arrangement of knots (their local intensification or multiplication).

When knots are closely spaced around a certain point, the B-spline basis functions are sharper and more localized around it. This increases the resolution and flexibility of the basis functions in that area and allows to capture more detail where the solution has high gradients. In other words, a local refinement is preserved.

Increasing the multiplicity of a knot (i.e. repeating the same knot value) changes the behavior of the basis functions by reducing the continuity at that knot. This is important when solving problems with the physical discontinuities, e.g. concentrated loads.

The B-spline functions with $q = 2$ and different knot vectors (uniform, concentrated and duplicated) are shown in Fig. 2.

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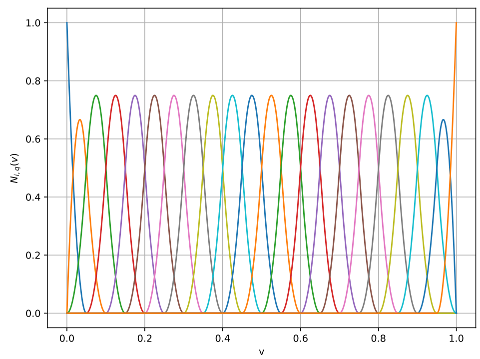
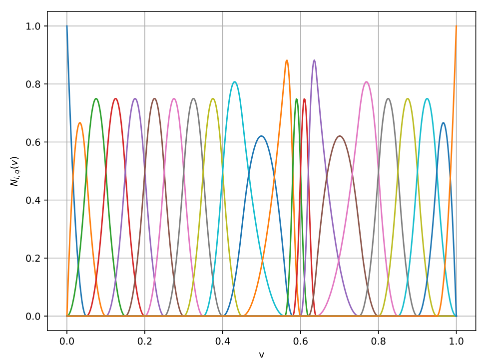
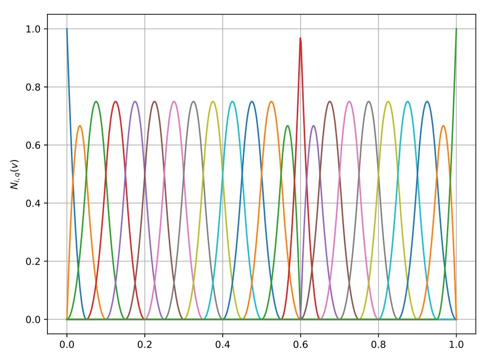


(a)



(b)



(c)

Fig. 2: B-spline basis functions for $q = 2$ and knot vector: a) uniform, b) concentrated around point 0.6, c) duplicated at point 0.6.

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Finally, the PIES method is solved using the collocation method [17]. The arrangement of collocation points is also essential. However, the authors have studied the influence of collocation points' arrangement and number (M) on the solutions in the previous research. Therefore, in this paper, both parameters are fixed, and the analysis focuses on the properties of the B-spline function.

5 Numerical examples

5.1 Example 1 - half-space with a concentrated boundary condition

The first example concerns a half-space with a concentrated Neumann boundary condition given by the Dirac function. The problem is modeled by the Laplace equation and is presented in Fig. 3.

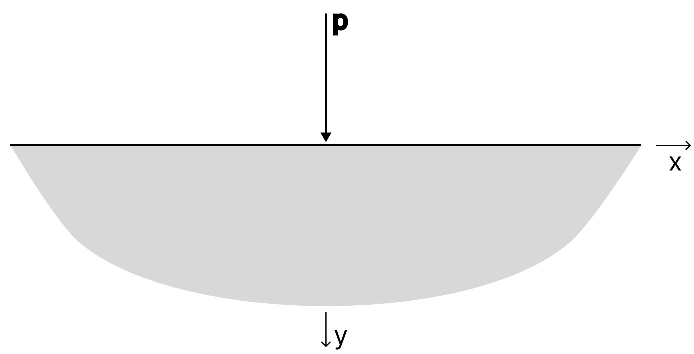

Fig. 3: A half-space with a concentrated boundary condition.

As a result of applying PIES presented in [1] to the problem being solved, an analytical solution on the boundary is obtained

$$u(x) = \frac{1}{\pi} ln(\frac{1}{\sqrt{x^2}}). \tag{7}$$

Then, PIES was solved numerically for the case where the half-space is bounded by a large region modeled using Bezier curves of the first degree. Chebyshev polynomials and B-spline functions were used to approximate the solutions using (4) with $M = 10$. B-spline function properties are $q = 2$ and $\boldsymbol{V} = \{-0.5, -0.5, -0.5, -0.15, -0.013, 0, 0, 0, 0.013, 0.15, 0.5, 0.5, 0.5\}$. Due to the expected singularity of the solution at point $x = 0$, the middle knot has been duplicated to reduce continuity. The obtained results in a certain region around the singularity are compared with the analytical solution (7) and presented in Fig. 4.

As shown in Fig. 4, solutions obtained by PIES with B-spline functions are much more accurate than those generated using Chebyshev polynomials. It can

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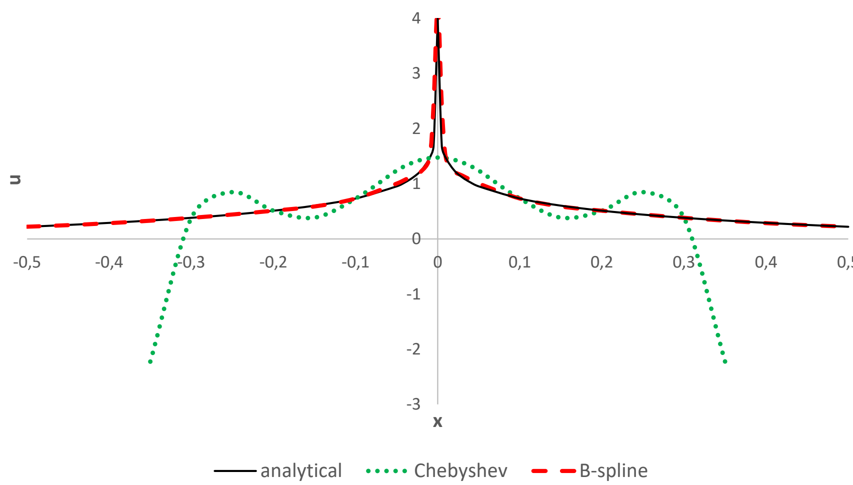

Fig. 4: Comparison of solutions u on the boundary.

be seen that when polynomials are used to approximate a function with a singularity, oscillations appear. They are also far from the singular point because a singularity in one place affects the entire approximation. Due to its local structure, the B-spline can approximate the function on both sides of the singularity well, but it should be emphasized that the results strongly depend on knots. Therefore, they should be carefully selected. Fig. 5 presents solutions using various knot vectors $\boldsymbol{V1}, \boldsymbol{V2}, \boldsymbol{V3}$. The knot arrangement is given at the bottom of the figure (the knots' color corresponds to the solution's color). It should also be remembered that some knots are duplicated, which may not be visible in the figure. $q+1$ knots from the beginning and end of the approximation interval force the function to pass through the first and last control points. The middle knot duplication reduces continuity at that location to enable modeling problems with concentrated boundary conditions.

The solutions presented in Fig. 5 show that accurate considering the singularity requires concentrating knots in its vicinity, not only the multiplication of the middle knot. This is visible in the vector $\boldsymbol{V3}$ (green), where the knot at point $x = 0$ is multiplicated, but the neighboring knots are moved slightly away from the center. This results in a much lower accuracy of solutions exactly at the singular point (lower peak). On the other hand, the densification only in the center, while shifting the remaining knots to the ends (vector $\boldsymbol{V2}$, blue) causes disturbances in the broader range around the singular point and at the ends compared to the $\boldsymbol{V3}$ case.

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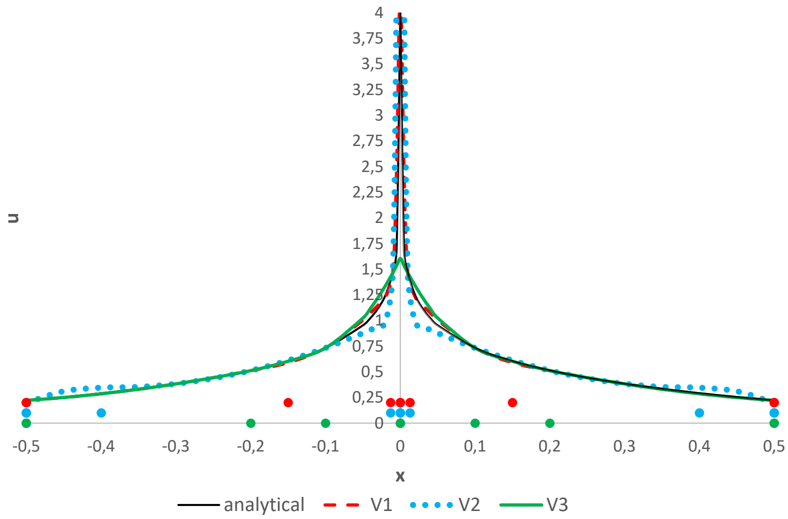

Fig. 5: Solutions u depending on various knot vectors.

5.2 Example 2 - elastic plate with a circular hole

The second example concerns the elastic plate with the circular hole subjected to the tensile load p at its ends (Fig. 6). Plane stress conditions are assumed with the following material properties $E = 200 \times 10^9 MPa$ and $v = 0.33$. Each segment is modeled by the Bezier curve (of the first or third degree).

Once again, for approximation of boundary solutions, two different basis functions (Chebyshev and B-spline) are applied in (4) with $M = 7$. Various q degrees and knots are analyzed for the B-spline function. The results for stresses σ_y at the part of the bottom boundary segment (around the hole) are obtained and compared with the existing analytical solution [18]

$$\sigma_y = \frac{p}{2}(2 + \frac{1}{r^2} + 3\frac{1}{r^4}), \tag{8}$$

where r is one of the polar coordinates and $r^2 = x^2 + y^2$. The result of the comparison together with the knots arrangement is presented in Fig. 7.
As seen in Fig. 7, the degree of the B-spline function plays an essential role in the accuracy of the obtained solutions. The best fit to the analytical results is obtained for $q = 4$, but it should be emphasized that even the solution for $q = 2$ is much more accurate than that obtained using Chebyshev polynomials. The average error of the solutions obtained using the B-spline function on the considered part of the segment decreased from 10.34% (for $q = 1$) to 2.27% ($q = 4$). For comparison, this error for Chebyshev polynomials is 11.89%.

Analyzing the knots arrangement, it is visible that the desired distribution is their more significant number in the singular region, which occurs near the hole.

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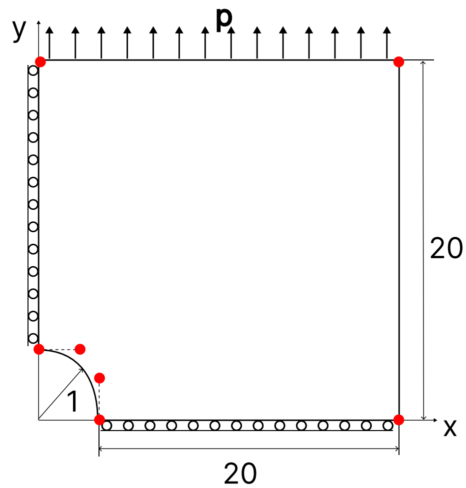

Fig. 6: The plate with a circular hole.

Once again, $q+1$ extreme knots are duplicated, which ensures that the function starts and ends precisely at the ends of the boundary segment.

6 Conclusions

This paper has derived and tested the approach for solving boundary problems with singular solutions. The PIES method combines the B-spline functions as basis functions in the approximation series. They require determining the degree and defining knots, which allows controlling the accuracy of the singular solutions by placing knots adaptively or increasing knot multiplicity.

Two examples were considered in the paper. The first one concerned a half-space with a boundary condition concentrated at a single point modeled by the Laplace equation. This example was chosen because of the possibility of obtaining an exact solution for comparison purposes. The second one is a typical elastic example, where we deal with a plate with a hole subjected to the tensile load. In both cases, the solutions are singular.

The results obtained from the numerical analysis show that using the B-spline as basis functions significantly increases the accuracy of modeling singular solutions. With an appropriately selected degree and knots, the errors of the solutions are minimized. The biggest challenge is the selection of appropriate knots, because their arrangement has a significant impact on the final results. It would be worth looking for optimal arrangements of knots using any optimization algorithm. Some metaheuristics can be used to optimize the form of the knot vector, but this requires a suitable objective function. The most straightforward would be to minimize the difference between the analytical and numerical solutions. However, developing an approach for cases where the exact solution does

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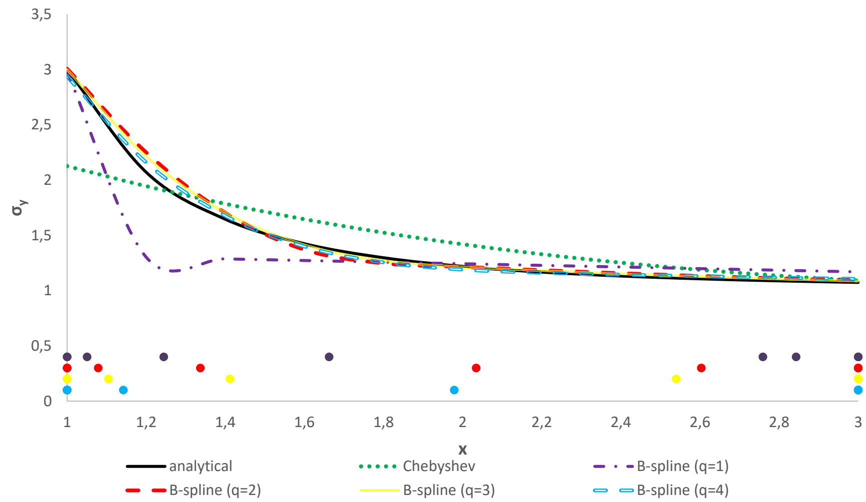

Fig. 7: Solutions σ_y for various degrees of B-spline functions.

not exist would be worth trying. Moreover, in this paper, some PIES properties are fixed (collocation points number and arrangement). Still, it would also be worth examining what relation between those properties and knots gives the best results. Again, the optimal relationship using optimization algorithms can be found. Finally, comparing the results obtained by PIES with other methods that use B-spline functions to approximate solutions also would be interesting.

Disclosure of Interests. The authors have no competing interests to declare that are relevant to the content of this article.

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