Numerical Results and Convergence of Some Inf-Sup Stable Elements for the Stokes Problem with Pressure Dirichlet Boundary Condition

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Abstract. For the Stokes problem with pressure Dirichlet boundary conditions, we propose an Enriched Mini element. For both the Mini element and the Enriched Mini element, we show that they are inf-sup stable. Unexpectedly, they yield wrong convergent finite element solutions for the singular velocity solution. On the contrary, the Taylor-Hood element, which is still inf-sup stable, gives correct convergence. However, how to analyze the convergence becomes open. We provide extensive numerical studies on the wrong convergence of the inf-sup stable Mini-type elements and the correct convergence of the inf-sup stable Taylor-Hood element and on the inf-sup stability constants.

Keywords: Stokes problem · Pressure Dirichlet boundary condition · Non H^1 velocity · Inf-sup stable element · Wrong convergence.

1 Introduction

The Mini element ([1,7,3]) is inf-sup stable for the Stokes problem. This element uses the nodal linear element enriched with one element bubble for each component of the velocity variable and the nodal linear element for the pressure variable. It gives an optimal convergent approximation of the velocity variable. Unexpectedly, we found that if the pressure variable has its own boundary condition on the part or the whole of the domain boundary and if the velocity variable is singular and does not belong to the Hilbert space H^1, the Mini element gives a wrong solution. The non H^1 singularity does not sound peculiar. In general, the Stokes problem lives with the Dirichlet integral, and the H^1 space is a natural solution space for the velocity variable. This is the case when the velocity variable is imposed with the Dirichlet boundary condition on the whole domain boundary. However, in practical situations (cf. [6]), the velocity variable may only have partial Dirichlet boundary condition while its tangential components have other partial Dirichlet boundary condition. As a supplement, the pressure variable accordingly has partial Dirichlet boundary condition. Under these boundary conditions, the velocity variable would not belong to the H^1 space (cf. [5], [3]). Following the theory in [8], we enrich the Mini element adding one degree of freedom in the interior of each elemental side locating on the domain boundary(just considering the two-dimensional problem). However, the Enriched Mini

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DOI: 10.1007/978-3-031-63783-4_2

element is inf-sup stable, but it still wrongly converges. We also develop a new general approach for proving the inf-sup stability of the Mini element for the Stokes equations with pressure Dirichlet boundary conditions. On the contrary, for the Taylor-Hood element [2,7,3], it generates correctly convergent approximations for the pressure Dirichlet boundary condition. The convergence is not optimal for smooth solutions, but a theoretical converge rate for the singular solution can be reached in the numerical results. This element is inf-sup stable, as is shown in [8]. Unexpectedly, it seems extremely difficult to give a convergence analysis, which then becomes an open problem. From the kernel coercivity and the inf-sup stability of the Taylor-Hood element, the convergence cannot be theoretically justified whenever the velocity solution does not belong to the H^1 space. For the Stokes problem with the velocity Dirichlet boundary condition on the whole domain boundary, as is well-known ([7], [3]), the convergence and the optimality of the Taylor-Hood element follow from the kernel coercivity and the inf-sup stability. The Taylor-Hood element has extensive applications, e.g., its application in the multiphysics problems([10]). The inf-sup stability plays a key role in the multigrid method and the preconditioning method([11], [12]) for the saddle-point systems of the Stokes equations and the vector Laplacian. Note that the Stokes equations (1) becomes the vector Laplacian when $\Gamma_2 = \partial\Omega$.

2 Stokes Problem, Mini Element and Taylor-Hood Element

Let $\Omega \subset \mathbb{R}^2$ be a simply-connected domain, with boundary $\partial\Omega = \bar{\Gamma}_1 \cup \bar{\Gamma}_2, \Gamma_1 \cap \Gamma_2 = \emptyset$. The Stokes problem reads as follows:

$$-\Delta \mathbf{u} + \nabla p = \mathbf{f}, \quad \operatorname{div} \mathbf{u} = 0 \quad \text{in } \Omega, \tag{1}$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_1, \quad \mathbf{n} \times \mathbf{u} = \mathbf{0}, \quad p = 0 \quad \text{on } \Gamma_2. \tag{2}$$

When $\Gamma_1 = \partial\Omega$, we require that $\int_\Omega p = 0$. Define $\mathbf{U} = \{\mathbf{v} \in H(\mathbf{curl}\,;\Omega) \cap H(\operatorname{div}\,;\Omega) : \mathbf{v}|_{\Gamma_1} = \mathbf{0}, \mathbf{n} \times \mathbf{v}|_{\Gamma_2} = \mathbf{0}\}$, $Q = \{q \in H^1(\Omega) : q|_{\Gamma_2} = 0\}$. Let $\mathscr{T}_h$ denote the shape-regular triangulation of Ω into triangles. A generic element $T \in \mathscr{T}_h$ has its diameter h_T; $h := \max_{T \in \mathscr{T}_h} h_T$. Let $\mathscr{P}_\ell$ be the space of polynomials of total degree less than or equal to the integer $\ell \geq 0$. Let $\lambda_1, \lambda_2, \lambda_3$ denote the three shape functions of $\mathscr{P}_1(T)$ on a generic element T. Define the element bubble $b_T := \lambda_1\lambda_2\lambda_3$, and $\mathscr{P}_1^+(T) := \operatorname{span}\{\lambda_1, \lambda_2, \lambda_3, b_T\}$. Then, define $\mathbf{U}_h^{\text{Mini}} - Q_h$ the Mini element and $\mathbf{U}_h^{\text{T-H}} - Q_h$ the Taylor-Hood element, respectively,

$$\mathbf{U}_h^{\text{Mini}} = \{\mathbf{v}_h \in \mathbf{U} \cap (H^1(\Omega))^2 : \mathbf{v}_h|_T \in (\mathscr{P}_1^+(T))^2, \forall T \in \mathscr{T}_h\}, \tag{3}$$

$$\mathbf{U}_h^{\text{T-H}} = \{\mathbf{v}_h \in \mathbf{U} \cap (H^1(\Omega))^2 : \mathbf{v}_h|_T \in (\mathscr{P}_2(T))^2, \forall T \in \mathscr{T}_h\}, \tag{4}$$

$$Q_h = \{q_h \in Q : q_h|_T \in \mathscr{P}_1(T), \forall T \in \mathscr{T}_h\}. \tag{5}$$

The finite element problem reads as follows: Find $\mathbf{u}_h \in \mathbf{V}_h$ and $p_h \in Q_h$ such that

$$\begin{cases} a_h(\mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p_h) = (\mathbf{f}, \mathbf{v}_h) & \forall \mathbf{v}_h \in \mathbf{V}_h, \\ b(\mathbf{u}_h, q_h) = 0 \quad \forall q_h \in Q_h, \end{cases} \tag{6}$$

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DOI: 10.1007/978-3-031-63783-4_2

where $\mathbf{V}_h$ stands for either $\mathbf{U}_h^{\mathrm{Mini}}$ or $\mathbf{U}_h^{\mathrm{T-H}}$, and $a_h(\mathbf{u},\mathbf{v}) = (\mathbf{curl}\,\mathbf{u},\mathbf{curl}\,\mathbf{v}) + \sum_{T\in\mathscr{T}_h} h_T^2(\mathrm{div}\,\mathbf{u},\mathrm{div}\,\mathbf{v})_{0,T}$ and $b(\mathbf{v},q) = (\mathbf{v},\nabla q)$. The L^2 inner product is denoted by $(\cdot,\cdot)_{0,D}$ and when $D = \Omega$, the subscripts $0, D$ are dropped. It is necessary to adopt the bilinear form $a_h(\cdot,\cdot)$ instead of the classical Dirichlet integral bilinear form $a_1(\mathbf{u},\mathbf{v}) = (\nabla\mathbf{u},\nabla\mathbf{v})$ and the curl-div bilinear form $a_2(\mathbf{u},\mathbf{v}) = (\mathbf{curl}\,\mathbf{u},\mathbf{curl}\,\mathbf{v}) + (\mathrm{div}\,\mathbf{u},\mathrm{div}\,\mathbf{v})$. Both $a_1(\cdot,\cdot)$ and $a_2(\cdot,\cdot)$ always lead to wrong convergent approximations for the singular velocity solution. The velocity solution $\mathbf{u}\in\mathbf{U}$, but $\mathbf{U}\subset(H^r(\Omega))^2$ for some $0<r<1$, i.e., $\mathbf{U}\not\subset(H^1(\Omega))^2$ unless Ω is smooth enough. The H^1-conforming element cannot be dense in $\mathbf{U}$ with respect to $a_1(\cdot,\cdot)$ or $a_2(\cdot,\cdot)$. But, the density holds for $a_h(\cdot,\cdot)$, cf. [4].

3 Kernel Coercivity, Inf-Sup Stability, Convergence and Open Problem

The bilinear form $a_h(\cdot,\cdot)$ already induces a norm on $\mathbf{V}_h$, and in particular, from $a_h(\mathbf{v}_h,\mathbf{v}_h) = 0$ we have $\mathbf{curl}\,\mathbf{v}_h = \mathbf{0}, \mathrm{div}\,\mathbf{v}_h = 0, \mathbf{v}_h\times\mathbf{n} = \mathbf{0}$, and as a result, $\mathbf{v}_h\equiv\mathbf{0}$. From [8], we have the kernel coercivity with respect to the L^2-norm:

$$a_h(\mathbf{v}_h,\mathbf{v}_h) \geq C||\mathbf{v}_h||_0^2 \quad \forall\mathbf{v}_h\in\mathbf{V}_h. \tag{7}$$

On the other hand, only for the Taylor-Hood element $\mathbf{V}_h := \mathbf{U}_h^{\mathrm{T-H}}$, the inf-sup stability was proven in [8], for some constant $\mu>0$ independent of h,

$$\sup_{\mathbf{0}\neq\mathbf{v}_h\in\mathbf{V}_h} \frac{b(\mathbf{v}_h,q_h)}{||\mathbf{v}_h||_1} \geq \mu||q_h||_0 \quad \forall q_h\in Q_h. \tag{8}$$

Here $||\cdot||_1$ is the norm of the Hilbert space $H^1(\Omega)$, i.e., $||v||_1^2 = ||v||_0^2 + ||\nabla v||_0^2$, and $||\cdot||_0$ denotes the L^2 norm. Note that the inf-sup stability holds with respect to $||\cdot||_1$, and of course it holds with respect to $|||\cdot|||_h^2 := ||\cdot||_0^2 + ||\cdot||_{a_h}^2$, with $||\cdot||_{a_h}^2 := a_h(\cdot,\cdot)$. Regarding the Mini element, following the local argument in [7,3], it is not difficult to show the weaker inf-sup stability: where $|q|_{1,h}^2 := \sum_{T\in\mathscr{T}_h} h_T^2||\nabla q||_{0,T}^2$,

$$\sup_{\mathbf{0}\neq\mathbf{v}_h\in\mathbf{U}_h^{\mathrm{Mini}}} \frac{b(\mathbf{v}_h,q_h)}{||\mathbf{v}_h||_1} \geq C|q_h|_{1,h} \quad \forall q_h\in Q_h. \tag{9}$$

In order to obtain (8) following the theory in [8], we enrich the Mini element in the following way: for every elemental side $F\subset\partial T$ locating on Γ_2, we add one degree of freedom in its midpoint. The base function can be chosen as the midpoint shape function of the quadratic element $\mathscr{P}_2(T)$. Denote this modified Mini element by $\mathbf{U}_h^{\mathrm{Mini},\Gamma_2}$. Below, we call $\mathbf{U}_h^{\mathrm{Mini},\Gamma_2}$ the Enriched Mini element. Then, from the nontrivial argument in [8], we conclude that (8) holds for $\mathbf{V}_h := \mathbf{U}_h^{\mathrm{Mini},\Gamma_2}$.

Theorem 1. *The Enriched Mini element satisfies the inf-sup stability* (8).

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DOI: 10.1007/978-3-031-63783-4_2

However, the theory in [8] does not cover the Mini element, wherein whether the inf-sup stability (8) holds or not for the Mini element is stated as an open problem. Below, we give an approach to complete this problem. This approach uses the weaker inf-sup stability (9) and the regular-singular decomposition [9]. It is new, different from the local stability argument (cf., [1,3,7]) and also different from the argument in [8].

Theorem 2. *The Mini element satisfies the inf-sup stability* (8).

Proof. From [9], on a Lipschitz polygon Ω, for $\mathbf{w} \in H_0(\mathrm{div}\,; \Omega) \cap H(\mathbf{curl}\,; \Omega)$, we have the regular-singular decomposition $\mathbf{w} = \mathbf{w}^{\mathrm{reg}} + \nabla p^{\mathrm{sing}}$, where $\mathbf{w}^{\mathrm{reg}} \in H_0(\mathrm{div}\,; \Omega) \cap (H^1(\Omega))^2, p^{\mathrm{sing}} \in H^1(\Omega)/\mathbb{R}$. Moreover, $||\mathbf{w}^{\mathrm{reg}}||_1 \leq C(||\mathbf{curl}\,\mathbf{w}||_0 + ||\mathrm{div}\,\mathbf{w}||_0)$. Put $\mathbf{v} := (w_2, -w_1)$ and $\mathbf{v}^{\mathrm{reg}} := (w_2^{\mathrm{reg}}, -w_1^{\mathrm{reg}})$. Then $\mathbf{v}, \mathbf{v}^{\mathrm{reg}} \in H_0(\mathbf{curl}\,; \Omega) \cap H(\mathrm{div}\,; \Omega)$, $\mathbf{v} = \mathbf{v}^{\mathrm{reg}} + \mathbf{curl}\, p^{\mathrm{sing}}$ and $||\mathbf{v}^{\mathrm{reg}}||_1 \leq C(||\mathbf{curl}\,\mathbf{v}||_0 + ||\mathrm{div}\,\mathbf{v}||_0)$. Here consider the case $\Gamma_2 := \partial\Omega$ only. For any given $q_h \in Q_h$, introduce the problem: Find $\theta \in H_0^1(\Omega)$ such that $-\Delta\theta = q_h$ in Ω and $\theta = 0$ on $\partial\Omega$. Then, $\mathbf{v} := \nabla\theta$. We have $\mathbf{v} \in H_0(\mathbf{curl}\,; \Omega) \cap H(\mathrm{div}\,; \Omega)$, $-\mathrm{div}\,\mathbf{v} = q_h$, and we have the regular-singular decomposition $\mathbf{v} = \mathbf{v}^{\mathrm{reg}} + \mathbf{curl}\, p^{\mathrm{sing}}$. Since $\mathbf{v}^{\mathrm{reg}} \in (H^1(\Omega))^2$, from [3], it is not difficult to find $\mathbf{v}_h^{\mathrm{reg}} \in \mathbf{V}_h := \mathbf{U}_h^{\mathrm{Mini}}$ denoting the finite element interpolation such that

$$\left(\sum_{T \in \mathscr{T}_h} h_T^{-2} ||\mathbf{v}_h^{\mathrm{reg}} - \mathbf{v}^{\mathrm{reg}}||_{0,T}^2 \right)^{\frac{1}{2}} + ||\mathbf{v}_h^{\mathrm{reg}}||_1 \leq C||\mathbf{v}^{\mathrm{reg}}||_1, \tag{10}$$

where $||\mathbf{v}^{\mathrm{reg}}||_1 \leq C(||\mathbf{curl}\,\mathbf{v}||_0 + ||\mathrm{div}\,\mathbf{v}||_0) = C||\mathrm{div}\,\mathbf{v}||_0 = C||q_h||_0$. Now,

$$\sup_{\mathbf{0} \neq \mathbf{v}_h \in \mathbf{V}_h} \frac{b(\mathbf{v}_h, q_h)}{||\mathbf{v}_h||_1} \geq \frac{b(\mathbf{v}_h^{\mathrm{reg}}, q_h)}{||\mathbf{v}_h^{\mathrm{reg}}||_1} = \frac{b(\mathbf{v}^{\mathrm{reg}}, q_h)}{||\mathbf{v}_h^{\mathrm{reg}}||_1} + \frac{b(\mathbf{v}_h^{\mathrm{reg}} - \mathbf{v}^{\mathrm{reg}}, q_h)}{||\mathbf{v}_h^{\mathrm{reg}}||_1},$$

where, since $(\mathbf{curl}\, p^{\mathrm{sing}}, \nabla q_h) = 0$, there exists a constant $C_1 > 0$ such that

$$\frac{b(\mathbf{v}^{\mathrm{reg}}, q_h)}{||\mathbf{v}_h^{\mathrm{reg}}||_1} = \frac{b(\mathbf{v}, q_h)}{||\mathbf{v}_h^{\mathrm{reg}}||_1} = \frac{||q_h||_0^2}{||\mathbf{v}_h^{\mathrm{reg}}||_1} \geq C\frac{||q_h||_0^2}{||\mathbf{v}^{\mathrm{reg}}||_1} \geq C_1||q_h||_0,$$

and from (10), there exists $C_2 > 0$ such that $\dfrac{b(\mathbf{v}_h^{\mathrm{reg}} - \mathbf{v}^{\mathrm{reg}}, q_h)}{||\mathbf{v}_h^{\mathrm{reg}}||_1} \geq -C_2|q_h|_{1,h}$. Hence, it follows that $\sup_{\mathbf{0} \neq \mathbf{v}_h \in \mathbf{V}_h} \dfrac{b(\mathbf{v}_h, q_h)}{||\mathbf{v}_h||_1} \geq C_1||q_h||_0 - C_2|q_h|_{1,h}$. Combing the weaker inf-sup stability (9), we obtain the conclusion.

The above approach is general, covering any pair $\mathbf{V}_h$ and Q_h, so long as the weaker inf-sup stability (9) holds. However, this approach seems not be applicable to three-dimensional problems, because it relies on the relation between the ∇ operator and the **curl** operator in two dimensions while such relation does not hold any longer in three dimensions.

Open Problem The kernel coercivity (7) and the inf-sup stability (8) are not sufficient to guarantee the correct convergence if the exact velocity solution is singular. The numerical results show that both the Mini element and

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DOI: 10.1007/978-3-031-63783-4_2

the Enriched Mini element generate wrong approximations. The Taylor-Hood element yields a correctly convergent solution for the singular velcoity solution. The challenging issue is how to prove the convergence.

4 Numerical Results

We report the numerical results. The main purpose is to investigate the stability, the convergence and the error bound for the smooth and singular velocity solution, we always set the exact pressure $p := 0$; meanwhile, we only consider $\Gamma_2 = \partial\Omega$, i.e., $\mathbf{n} \times \mathbf{u} = \mathbf{0},\ p = 0 \quad \text{on } \partial\Omega$. In engineering applications, the pressure is very often continuous. Whenever Ω is nonsmooth with reentrant corners and edges, under the above boundary conditions, the velocity solution is usually singular, i.e., lying outside H^1. Choosing the L-shaped domain $\Omega = (-1,1)^2 \setminus ([0,1] \times [-1,0])$ and using the uniform mesh of triangles, we numerically study the following issues:

- The Mini element and the Enriched Mini element. (1) Correct and optimal convergence for smooth velocity solution. (2) Wrong convergence for singular velocity solution. (3) Inf-sup stability holds.
- The Taylor-Hood element. (1) Correct but suboptimal convergence for smooth velocity solution. (2) Correct convergence with the optimal rate the same as the regularity of the singular velocity solution.

Example 1: Smooth Velocity Solution.
Let the exact smooth velocity solution $\mathbf{u}$ be $u_1(x,y) = \sin(\pi y)\cos(\pi x)$, $u_2(x,y) = -\sin(\pi x)\cos(\pi y)$. The numerical results are reported in Table 1 and Table 2. The Mini element and the Enriched Mini element give the optimal convergence $O(h^2)$ for both the velocity and the pressure. The Taylor-Hood element gives the suboptimal convergence $O(h^2)$ for the velocity; the pressure shows super-convergence $O(h^3)$.

Table 1. Errors of velocity $||\mathbf{u} - \mathbf{u}_h||_0$

h	Mini element		Enriched Mini element		Taylor-Hood element	
	Error	Order	Error	Order	Error	Order
1/4	9.37e − 02	–	9.23e − 02	–	1.01e − 02	–
1/8	2.03e − 02	2.21	2.03e − 02	2.18	1.40e − 03	2.85
1/16	4.86e − 03	2.06	4.88e − 03	2.06	2.03e − 04	2.79
1/32	1.20e − 03	2.02	1.20e − 03	2.02	3.51e − 05	2.53
1/64	3.00e − 04	2.00	3.00e − 04	2.01	7.46e − 06	2.23

Example 2: Singular Velocity.
Let the exact singular velocity $\mathbf{u}$ be $u_1(x,y) = -\frac{2}{3}\rho^{-1/3} \cdot \sin\left(\frac{\theta}{3}\right)$, $u_2(x,y) = \frac{2}{3}\rho^{-1/3} \cdot \cos\left(\frac{\theta}{3}\right)$, where (ρ,θ) stands for the polar coordinates originating at the origin, with $\rho = \sqrt{x^2+y^2}, \tan\theta = y/x$. The regularity of $\mathbf{u}$ is $2/3 - \epsilon$ for any

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DOI: 10.1007/978-3-031-63783-4_2

Table 2. Errors of pressure $||p - p_h||_0$

h	Mini element		Enriched Mini element		Taylor-Hood element	
	Error	Order	Error	Order	Error	Order
1/4	4.97e − 02	–	5.23e − 02	–	2.07e − 04	–
1/8	2.40e − 02	1.05	2.47e − 02	1.08	6.42e − 05	1.69
1/16	7.09e − 03	1.76	7.18e − 03	1.78	8.87e − 06	2.86
1/32	1.86e − 03	1.93	1.87e − 03	1.94	9.91e − 07	3.16
1/64	4.71e − 04	1.98	4.73e − 04	1.98	9.97e − 08	3.31

small $\epsilon > 0$. The numerical results are reported in Table 3 and Table 4. Both the Mini element and the Enriched Mini element give the wrong approximations for both the velocity and the pressure. The Taylor-Hood element gives a correctly convergent approximation, with a correct convergence rate of about $O(2/3)$. The pressure also converges with super-convergence $O(h)$.

Table 3. Errors of velocity $||\mathbf{u} - \mathbf{u}_h||_0$

h	Mini element		Enriched Mini element		Taylor-Hood element	
	Error	Order	Error	Order	Error	Order
1/4	3.29e − 01	–	5.63e − 01	–	2.33e − 01	–
1/8	3.31e − 01	-0.01	5.91e − 01	-0.07	1.50e − 01	0.64
1/16	3.87e − 01	-0.23	6.44e − 01	-0.12	9.12e − 02	0.71
1/32	4.71e − 01	-0.28	7.01e − 01	-0.12	5.31e − 02	0.78
1/64	5.61e − 01	-0.25	7.48e − 01	-0.09	3.07e − 02	0.79

Table 4. Errors of pressure $||p - p_h||_0$

h	Mini element		Enriched Mini element		Taylor-Hood element	
	Error	Order	Error	Order	Error	Order
1/4	3.87e − 01	–	4.97e − 01	–	1.28e − 01	–
1/8	4.31e − 01	-0.15	5.41e − 01	-0.12	9.09e − 02	0.49
1/16	4.81e − 01	-0.16	5.73e − 01	-0.08	5.65e − 02	0.68
1/32	5.23e − 01	-0.12	5.90e − 01	-0.04	3.09e − 02	0.87
1/64	5.53e − 01	-0.08	5.96e − 01	-0.02	1.54e − 02	1.01

Example 3: More Singular Velocity and Singular Data.

Let a more singular velocity $\mathbf{u}$ be given with $u_1(x, y) = -\frac{1}{2}\rho^{-1/2} \cdot \sin\left(\frac{\theta}{2}\right)(x+1)(y+1)$ and $u_2(x, y) = \frac{1}{2}\rho^{-1/2} \cdot \cos\left(\frac{\theta}{2}\right)(x+1)(y+1)$. The regularity of $\mathbf{u}$ is $1/2-\epsilon$ for any small $\epsilon > 0$. The right-hand sides $\mathbf{f}$, $g := \operatorname{div}\mathbf{u}$ and the boundary data $\chi := \mathbf{t} \cdot \mathbf{u}$ are all much more singular: they are not L^2 functions; they belong to some negative fractional order Sobolev spaces. The numerical results are reported in Table 5 and Table 6. Likewise, both the Mini element and the Enriched Mini

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DOI: 10.1007/978-3-031-63783-4_2

element do not converge, while the Taylor-Hood still gives a convergence with a rate slightly higher than the theoretical rate of about 1/2.

Table 5. Errors of velocity $||\mathbf{u} - \mathbf{u}_h||_0$

h	Mini element		Enriched Mini element		Taylor-Hood element	
	Error	Order	Error	Order	Error	Order
1/4	$4.68e-01$	–	$8.43e-01$	–	$3.61e-01$	–
1/8	$4.96e-01$	-0.08	$9.59e-01$	-0.19	$2.62e-01$	0.47
1/16	$6.29e-01$	-0.34	$1.15e+00$	-0.26	$1.82e-01$	0.53
1/32	$8.50e-01$	-0.44	$1.39e+00$	-0.28	$1.21e-01$	0.58
1/64	$1.13e+00$	-0.41	$1.67e+00$	-0.26	$8.09e-02$	0.58

Table 6. Errors of pressure $||p - p_h||_0$

h	Mini element		Enriched Mini element		Taylor-Hood element	
	Error	Order	Error	Order	Error	Order
1/4	$4.91e-01$	–	$6.53e-01$	–	$1.85e-01$	–
1/8	$6.30e-01$	-0.36	$8.28e-01$	-0.34	$1.48e-01$	0.32
1/16	$7.97e-01$	-0.34	$1.01e+00$	-0.28	$1.04e-01$	0.51
1/32	$9.81e-01$	-0.30	$1.19e+00$	-0.23	$6.39e-02$	0.70
1/64	$1.18e+00$	-0.26	$1.37e+00$	-0.21	$3.57e-02$	0.84

Computation on Inf-Sup Constant: (8).

We report the inf-sup constants in (8) for the Mini element, the Enriched Mini element, and the Taylor-Hood element. From Table 7 and Table 8, the inf-sup constants are bounded from below as h tends to zero.

Table 7. Inf-sup constant with norm $||\cdot||_1$

h	Mini element	Enriched Mini element	Taylor-Hood element
	μ	μ	μ
1/4	0.5107264	0.5151383	0.9276922
1/8	0.4334440	0.4349372	0.9132119
1/16	0.4011601	0.4014494	0.9039361
1/32	0.3910563	0.3911005	0.8980655
1/64	0.3882729	0.3882791	0.8943652

Conclusion. The Stokes inf-sup stable elements such as Mini elements and Taylor-Hood elements under the no-slip velocity boundary condition can still be inf-sup stable under the pressure Dirichlet boundary condition, as proven by a new theory developed in this paper and confirmed by the numerical results

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DOI: 10.1007/978-3-031-63783-4_2

Table 8. Inf-sup constant with norm $|||\cdot|||_h$

h	Mini element	Enriched Mini element	Taylor-Hood element
	μ	μ	μ
1/4	0.8319936	0.8364393	2.0150314
1/8	0.7974487	0.7987991	2.6165266
1/16	0.7383949	0.7391324	3.1131941
1/32	0.6924892	0.6927582	3.3252386
1/64	0.6637092	0.6637771	3.3637365

provided. However, numerical examples of singular and non-H^1 velocity studied have shown the wrong convergence of the Mini-type elements. They have also shown the correct convergence (albeit suboptimal) of the Taylor-Hood elements; but how to prove the convergence of the Taylor-Hood elements for singular velocity is open.

Acknowledgements. The authors were partially supported by the National Natural Science Foundation of China (No. 12371371,12261160361,11971366).

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To cite this paper please use the final published version:
DOI: 10.1007/978-3-031-63783-4_2