Fuzzy solutions of boundary problems using interval parametric integral equations system

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Abstract. This paper investigated the possibility of obtaining fuzzy solutions to boundary problems using the interval parametric integral equations system (IPIES) method. It focused on the IPIES method because, thanks to the analytical modification of the boundary integral equations (BIE), it does not require classical discretization. In this method, an original modification of directed interval arithmetic was also proposed. Solutions obtained using classical and directed interval arithmetic (known from the literature) were also presented for comparison. The extension of the IPIES method (to obtain fuzzy solutions) was to divide the fuzzy number into α-cuts (depending on the assumed confidence level). Then, such α-cuts were represented as interval numbers. Preliminary tests were carried out in which the influence of boundary condition uncertainty on fuzzy solutions (obtained using IPIES) was investigated. The analysis of solutions was presented on examples described by Laplace's equation. The accuracy verification of the fuzzy PIES solutions required a modification of known, exactly defined analytical solutions. They were defined using intervals and calculated using appropriate interval arithmetic in α-cuts to obtain fuzzy analytical solutions finally. The research showed the high accuracy of fuzzy solutions obtained using IPIES and confirmed the high potential of the method in obtaining such solutions.

Keywords: boundary problems · uncertainty · fuzzy solutions · interval arithmetic

1 Introduction

The wide application of computer simulation in practice shows that defining input data exactly (by real numbers) significantly limits and idealizes reality. Practically, these data are always given with some uncertainty resulting from experimental data or measurement errors. One of the more intuitive ways to model uncertainty is to use interval numbers. In the boundary problems, they have been used in the interval finite element method (IFEM) [1] and interval boundary element method (IBEM) [2]. However, most of the IFEM or IMEB

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DOI: 10.1007/978-3-031-36030-5_47

research focuses on boundary conditions or various parameters defined uncertainly. Researchers often omit the problem of modelling the uncertainty of the boundary shape by interval coordinates of points. Such modelling in IFEM and IBEM is troublesome due to the necessity of interval discretization. As a result of a significant increase in the number of interval input data, solutions are overestimated and useless in practice.

To solve the problem (by significantly reducing the amount of interval input data), the method of interval parametric integral equations system (IPIES) was proposed [3, 4]. The method's main advantage is the unnecessity of classical discretization [5, 6]. The functions that model the boundary's shape are included directly in the mathematical formalism of PIES. As a result, a small amount of input data is required to model the shape of the boundary and boundary conditions. This significantly reduces the number of equations in the system to be solved, which reduces the number of calculations, shortens the time and reduces the required computer resources. The previous studies [3, 4] proved insignificant overestimations and high accuracy of the interval solutions obtained by IPIES.

The paper presents the IPIES application to obtain fuzzy solutions to the boundary problem. The fuzzy set theory [7, 8] is another way that can be used to define the uncertainty of boundary problems. The advantage (in comparison with intervals) is obtaining additional information about the behaviour of the solutions inside the interval bounds. As in the previous methods of uncertainty modelling, the fuzzy finite element method (FFEM) [9] and the fuzzy boundary element method (FBEM) [10] can also be found in the literature.

2 Interval Parametric Integral Equations System

Including uncertainly defined boundary conditions in the PIES requires defining interval boundary functions. Therefore, the solution on the boundary (of the problem modelled by the Laplace equation) can be obtained by solving the interval PIES defined as follows:

$$0.5\boldsymbol{u}_l(s_1) = \sum_{j=1}^{n} \int_{\widehat{s}_{j-1}}^{\widehat{s}_j} \left\{ U^*_{lj}(s_1, s)\boldsymbol{p}_j(s) - P^*_{lj}(s_1, s)\boldsymbol{u}_j(s) \right\} J_j(s)ds, \tag{1}$$

where $l = 1, 2, ..., n$, and $\widehat{s}_{l-1} \leq s_1 \leq \widehat{s}_l$, $\widehat{s}_{j-1} \leq s \leq \widehat{s}_j$. The $\widehat{s}_{l-1}, \widehat{s}_{j-1}$ are the beginnings, and the $\widehat{s}_l, \widehat{s}_j$ are endings of the boundary segments exactly defined in a parametric coordinate system. The function $J_j(s)$ is the Jacobian to the segment of the curve S_m (where $m = j, l$).

The functions $\boldsymbol{p}_j(s)$, $\boldsymbol{u}_j(s)$ are interval parametric boundary functions on individual segments S_j of the boundary. One will be given as uncertainly defined (interval) boundary conditions, while the other will be searched for in the numerical solution of the interval PIES. The paper assumes an exactly defined boundary shape, so the kernels are defined classically:

$$U^*_{lj}(s_1, s) = \frac{1}{2\pi} \ln \frac{1}{[\eta_1^2 + \eta_2^2]^{0.5}}, P^*_{lj}(s_1, s) = \frac{1}{2\pi} \frac{\eta_1 n_j^{(1)}(s) + \eta_2 n_j^{(2)}(s)}{\eta_1^2 + \eta_2^2}, \tag{2}$$

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where $n_j^{(1)}(s), n_j^{(2)}(s)$ are components of the normal vector $\boldsymbol{n}$ to the boundary segment S_j. Kernels allow for the analytical inclusion in its mathematical formalism of the boundary shape by appropriate relations between segments $\eta_1 = S_l^{(1)}(s_1) - S_j^{(1)}(s), \eta_2 = S_l^{(2)}(s_1) - S_j^{(2)}(s)$.

Since the direct application of interval arithmetic known from the literature caused significant overestimations, it was decided to propose a modification of directed interval arithmetic for calculations in the above method [3, 4]. It consists of shifting the operations to the positive semi-axis in multiplication.

3 Fuzzy solutions to boundary problems

The interval numbers give only the values of a certain set's lower and upper bounds. So, researchers became interested in ways that also define the interior of such a set. In the fuzzy set theory [7, 8], it was found that the human ability to make the right decisions decreases due to the appearance of uncertainty in more complex systems. Such uncertainty can be easily expressed in words, i.e. using a linguistic variable, which can be fuzzified and defined by a fuzzy set. It is a set without clearly defined boundaries. The values inside have an additional function determining their degree of belonging to the set. Therefore, the fuzzy set A is represented by the function $\mu_A(x)$ called the membership function:

$$A = [(x, \mu_A(x))|x \in R, \mu_A(x) \in [0, 1]]. \quad (3)$$

Corespondingly, $\mu_A(x) = 1$ it means that x is in the set A, while $\mu_A(x) = 0$ means that x does not belong to this set. The shape of this function depends on the fuzzification method used. The best-known types of membership functions are the triangular and Gaussian functions. One special kind of fuzzy set is a fuzzy number. This term denotes a special case of a convex, normalized fuzzy set with a continuous membership function. An example is a triangular fuzzy number (TFN), a fuzzy set with a triangular membership function. In a simplified form, it is reduced to the L-R representation [8], where the fuzzy number is represented as $x = (m, a, b)$ (Fig. 1).

$$\mu_A(x) = \begin{cases} 0 & \text{for } x \le a, \\ \frac{x-a}{m-a} & \text{for } a < x \le m, \\ 1 & \text{for } x = m, \\ \frac{b-x}{b-m} & \text{for } m < x \le b, \\ 0 & \text{for } x \ge b, \end{cases}$$

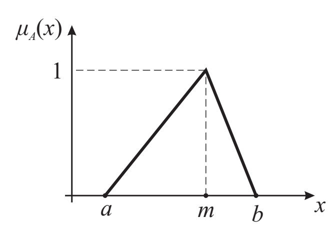

Fig. 1. A triangular fuzzy number

Obtaining fuzzy solutions with the interval PIES was to use a simplified notation of a fuzzy number using α-cuts. This method divides the membership

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function into certain levels called α-cuts. Each defines an interval in which the degree of values membership is greater than the given value α. So, the membership function can be defined using interval values: $\boldsymbol{x}_\alpha = [\underline{x}, \overline{x}]_\alpha$ easily. Respectively $\underline{x}$ is the smallest, and $\overline{x}$ is the largest value whose degree of membership is greater than or equal to α. A general schema of the application of fuzzy logic in modelling and simulating boundary problems is shown in Fig. 2.

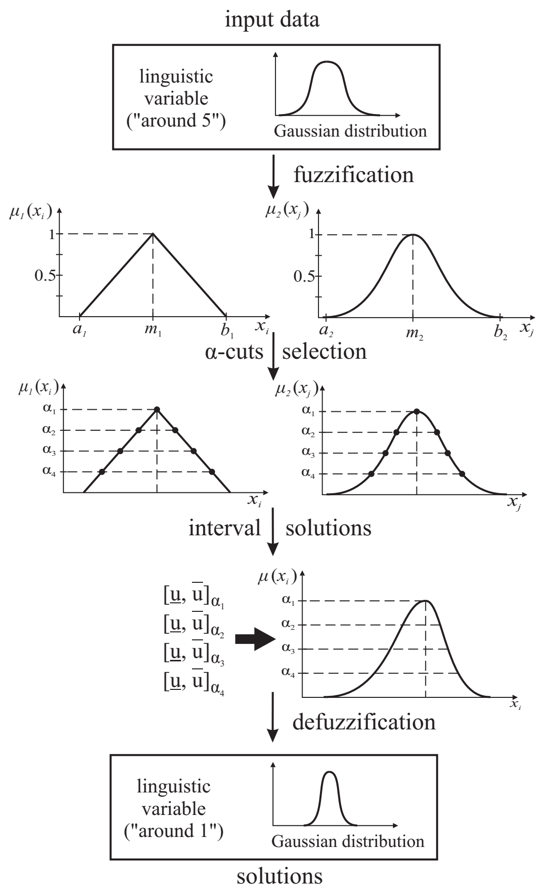

Fig. 2. A general schema of the application of fuzzy logic in modelling and simulating boundary problems.

The first stage is to define the input data with uncertainty. It can be a linguistic value, a real number, or a probability distribution. However, the most important thing is determining the appropriate way of fuzzification for a given variable. Fuzzification methods include intuition, reasoning, genetic algorithms, neural networks, induction or statistical distribution. After the fuzzification, the

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obtained data is defined as fuzzy numbers with the corresponding membership functions.

The next step is to divide the defined membership functions into the appropriate number of α-cuts for all uncertainly defined input data. The amount of information transferred to the simulation will depend on their number, which affects the amount of information obtained as a solution. Each α-cut is defined as an interval number. Interval solutions are calculated for α-cuts using appropriate interval arithmetic.

Finally, interval solutions are obtained for each α-cut. Therefore, obtaining the solution membership function is enough to set the interval solutions in the appropriate α-cuts (as shown in Fig. 2 at the "interval solutions" step). Finally, the defuzzification process on the membership function allows obtaining solutions as a probability distribution or an exact value.

4 Tests

The strategy of modelling and solving problems defined with uncertainty will be tested and analyzed on the example of problems described by Laplace's equation. The mentioned strategy (Fig. 2) using the method of IPIES [3, 4] from formula (3) was implemented as a computer program. Defining the uncertainty of all input data at once significantly reduces the possibility of identifying the causes of possible overestimations. Therefore, the preliminary research in this paper was limited to the verification of boundary conditions defined by fuzzy numbers.

Example 1. Square domain - constant fuzzy boundary conditions

The first elementary example was analyzed to enable the comparison of the proposed uncertainty modelling strategy [3, 4] with the classical [11] and directed [12] interval arithmetic known from the literature. The problem is shown in Fig. 3. One of the four segments was defined with the boundary condition by triangular fuzzy number $u_1 = (100, 95, 105)$. α-cuts were considered for the membership function $\mu(x) = 0, 0.1, 0.2, \ldots, 0.9, 1$.

The analytical solution of a similar problem with a precisely defined boundary condition $u = 100$ can be defined as $u = 100y$. Therefore, the modified interval analytical solution for the uncertainly defined problem can be presented as $u = [95, 105]y$. Interval solutions defined in this way (multiplication only by real number) reach the same values regardless of the interval arithmetic used. The interval solutions obtained on the α-cuts were saved as membership functions to obtain the final fuzzy analytical solution. Apart from the u_1 value, the boundary condition has been precisely defined (Fig. 3) to enable direct comparison between the IPIES and the interval analytical solutions.

Fuzzy PIES solutions (obtained using IPIES) and fuzzy analytical solutions (obtained using interval analytical solutions) are shown in Fig. 4. The solutions obtained using the proposed strategy (Fig. 4a) are almost equal to the proposed fuzzy analytical solutions (Fig. 4b). Additionally, solutions obtained directly applying for calculations in IPIES classical or directed interval arithmetic (known

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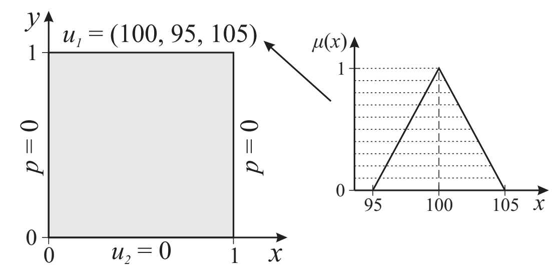

Fig. 3. Modelling example with fuzzy boundary conditions

from the literature) are presented. Even such an elementary example causes significant overestimations using classical interval arithmetic (Fig. 4d). In comparison, the directed interval arithmetic narrows the interval solutions' radii (Fig. 4c).

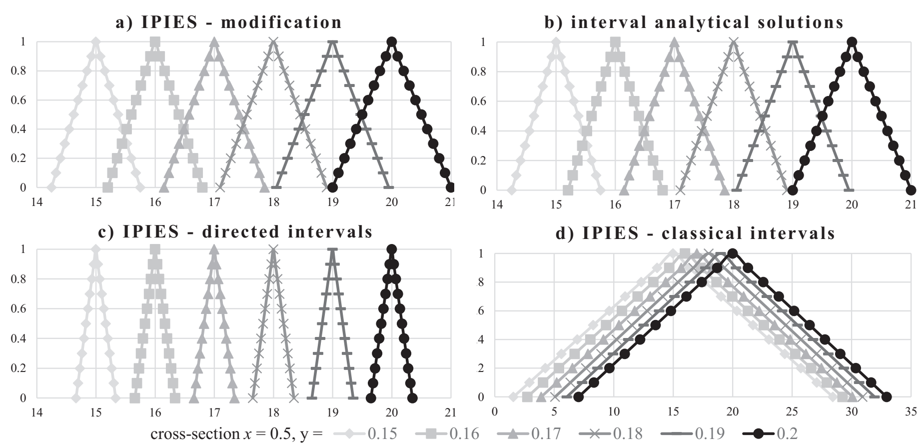

Fig. 4. Comparison of obtained fuzzy solutions in the domain.

Example 2. Complex area - a fuzzy function of boundary conditions

Another example is shown in Fig. 5. The analytical solution defined in the general form is $u = x^2 - 5y + x - y^2 + k$. Fuzzy solutions of the problem defined in this way (in the cross-section marked with black dots in Fig. 5), in analogous α-cuts as in the previous example, are shown in Fig. 6a. In addition, an exact analytical solutions were presented (for the middle value of the fuzzy number, i.e. $k = 50$). Additionally, an example with a constant fuzzy value of $k = (40, 50, 60)$ was assumed to enable direct comparison between fuzzy analytical solutions (Fig. 6c) and fuzzy PIES (Fig. 6b). The solutions obtained using fuzzy PIES are almost equal to those obtained using fuzzy analytical solutions.

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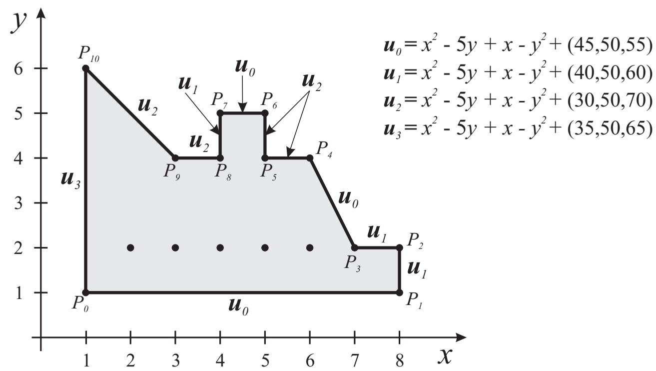

Fig. 5. Modelling example with fuzzy boundary conditions

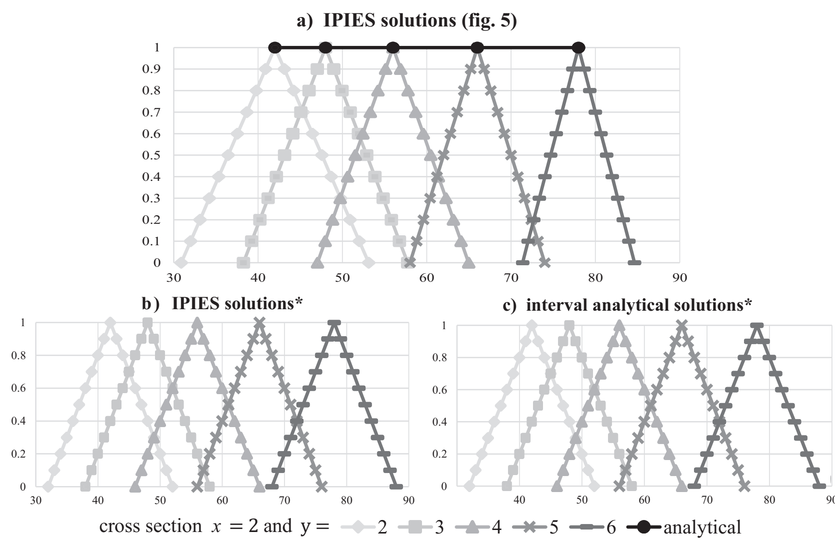

Fig. 6. Comparison of obtained fuzzy solutions in domain

5 Conclusions

The paper proposes the application of interval PIES to obtain fuzzy solutions to boundary problems. It was decided to focus only on the fuzzy boundary conditions to draw unambiguous conclusions. Such data were considered in subsequent α-cuts to allow IPIES application. After implementing the proposed algorithm, tests were carried out on the example of problems described by the Laplace equation. The disadvantages of the classical and directed interval numbers known from the literature and the advantages of the modification of directed interval arithmetic proposed in IPIES are highlighted in the first elementary example. Another more complex example also confirms the correctness of the solutions.

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Ultimately, as a result of the tests, the high potential of the IPIES method in obtaining fuzzy solutions was presented. In the subsequent research, more comprehensive tests are also planned for the uncertainty of the boundary shape and modifications of the algorithm to the other differential equations.

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