On the weak formulations of the Multipoint meshless FDM

Irena Jaworska

Cracow University of Technology, Poland
irena@cce.pk.edu.pl

Abstract. The paper discusses various formulations of the recently developed higher order Multipoint Meshless Finite Difference Method. The novel multipoint approach is based on raising the order of approximation of the unknown function by introducing additional degrees of freedom in stencil nodes, taking into account e.g. the right hand side of the considered differential equation. It improves the finite difference solution without increasing the number of nodes in an arbitrary irregular mesh. In general, the standard version of the Meshless (Generalized) FDM is based on the strong problem formulation. The extensions of the multipoint meshless FDM allow for analysis of boundary value problems posed in various weak formulations, including variational ones (Galerkin, Petrov-Galerkin), minimization of the energy functional, and the meshless local Petrov-Galerkin. Several versions of the multipoint method are proposed and examined. The paper is illustrated with some examples of the multipoint numerical tests carried out for the weak formulations and their comparison with those obtained for the strong formulation.

Keywords: Meshless FDM, Higher order approximation, Weak formulations, Multipoint method, Homogenization, Elastic-plastic problem

1 Introduction

The Finite Element Method (FEM) has been the most commonly applied method in the field of engineering computations, especially in computational mechanics, leading to solutions of practical engineering problems, such as fracture mechanics, elasticity, structural and fluid dynamics, many more. However, the problems dealing with for example the mesh distortion and frequent remeshing requirements are not efficient to solve by the FEM discretization. This was the motivation for the development of the alternative, so called meshless methods, prescribed by the set of nodal values only. Compared to the FEM, the arbitrarily irregularly distributed 'cloud of nodes' without any imposed structure can be easily modified [21, 22]. Therefore, this technique leads to greater flexibility and is more convenient and attractive to implement the adaptive process.

The local approximation of the unknown function in the meshless methods is performed around the nodes. A growing variety of meshless methods differ from each other in the process of construction of this local approximation, such as Moving Least

ICCS Camera Ready Version 2023
To cite this paper please use the final published version:
DOI: 10.1007/978-3-031-36027-5_41

Squares (MLS) [1] based methods: Meshless Finite Difference Method (MFDM/GFDM) [2], Diffuse Element Method [3], Element Free Galerkin [4], Finite Point Method [5], Meshless Local Petrov-Galerkin (MLPG) [6]; kernel methods: Smooth Particle Hydrodynamics [7, 8], Reproducing Kernel Particle Method [9], and Partition of Unity (PU) methods: *hp*-clouds [10], PU FEM [11], and many more.

Meshless methods can be classified in many different ways, among other by division into two categories: methods based on strong formulation and methods based on weak formulations. Many of these methods have also been developed in both forms. One of the oldest, as well as the most well-known meshless strong-form method is the Meshless Finite Difference Method (MFDM) [2, 12] which was later generalized into many meshless variants. Higher order method extensions, such as the Multipoint MFDM [13, 14] and MFDM based on the corrections terms [15] were developed for various formulations of the boundary value problems: strong (local), weak (global), and mixed (local-global) one. The multipoint meshless FDM posed in various weak formulations is briefly presented in this paper.

2 Multipoint problem formulation

The solution quality of boundary value problems (BVP), solved using meshless finite difference, may be improved through the application of two mechanisms. The first one is based on the mesh density increase, preferably using an adaptive (*h*-type) solution approach. The second mechanism is provided by rising the approximation order (*p*-type). Several concepts may be used in the last case [14 In the global formulation]. In this research, a return is being made to the old Collatz idea [16] of the multipoint FDM, which was combined with the MFDM to develop the new higher order multipoint meshless FDM.

The method is based on raising the order of approximation of the unknown function u by introducing additional degrees of freedom in the star nodes and using combinations of values of searched function and the right hand side of the MFD equations taken at all nodes of each MFD star.

Multipoint MFDM includes the following basic modifications and extensions: besides the local (strong), also the global (weak) formulations of BVP may be considered as well; arbitrary irregular meshes may be used; the moving weighted least squares (MWLS) approximation is assumed instead of the interpolation technique.

Let us consider the local (strong) formulations of boundary value problems for the *n*-th order PDE in the domain Ω

$$\begin{cases} \mathcal{L}u(P) = f(P), & \text{for} \quad P \in \Omega \\ \mathcal{G}u(P) = g(P), & \text{for} \quad P \in \partial\Omega \end{cases}, \tag{1}$$

or an equivalent global (weak) one involving integral with appropriate boundary conditions, where $\mathcal{L}, \mathcal{G}$ are respectively the *n*-th and *m*-th order differential operators.

In the FDM solution approach, the classical difference operator would be presented in the following form (assuming $u_i = u(P_i)$ and $f_i = f(P_i)$)

ICCS Camera Ready Version 2023
To cite this paper please use the final published version:
DOI: 10.1007/978-3-031-36027-5_41

$$\mathcal{L}u_i \approx Lu_i \equiv \sum_{j(i)} c_j u_j = f_i \quad \Rightarrow \quad Lu_i = f_i, \quad c_j = c_{j(i)} \; . \tag{2}$$

In the multipoint formulation, the MFDM difference operator Lu is obtained by the Taylor series expansion of unknown function u, including higher order derivatives, and using additional degrees of freedom at nodes. For this purpose one may use for example a combination of the right hand side values f of the given equation at each node of the stencil using arbitrarily distributed clouds of nodes:

$$\mathcal{L}u_i \approx Lu_i \equiv \sum_{j(i)} c_j u_j = \sum_{j(i)} \alpha_j f_j \quad \Rightarrow \quad Lu_i = Mf_i \; . \tag{3}$$

Here, j is the index of a node in a selected stencil (FD star), i is the index of the central node of the stencil, Mf_i is a combination of the equation right hand side values at the stencil associated to index i, f may present the value of the whole operator $\mathcal{L}u_i$ or a part of it only, such as a specific derivative $u^{(k)}$. In general, L may be referred to differential eqs, boundary conditions and integrand in the global formulation of BVP.

Two basic versions of the Multipoint MFDM – general and specific are considered [14]. The specific approach, presented above (3), is simpler and easier in implementation, but its application is more restricted, mainly to linear BVP. In the specific formulation, the values of the additional degrees of freedom are known. In the global formulation [13] , where $u^{(k)}$ derivative is assumed instead of f

$$\sum_{j(i)} c_j u_j = \sum_{j(i)} \alpha_j u_j^{(k)} \; , \tag{4}$$

they are sought. In each of these multipoint FD cases, one may usually obtain higher order approximation of the FD operators, using the same stencil, as needed to generate Lu_i in the classical FDM approach based on interpolation.

Such extended higher order MFDM approach also enables analysis of boundary value problems posed in various weak (global) formulations, including different variational formulations (Galerkin, Petrov-Galerkin), minimization of the energy functional, and MLPG. Various versions of the multipoint method in global formulations are briefly discussed.

3 Weak formulations of the multipoint MFDM approach

Besides the development of the multipoint MFDM for analysis of boundary value problems in the strong (local) formulation, the multipoint method was also extended to the weak (global) formulations.

The global formulation may be posed in the domain Ω in general as:

- *a variational principle*

$$b(u, v) = l(v), \qquad \forall \; v \in V \; , \tag{5}$$

where b is a bilinear functional dependent on the test function v and trial function u (solution of the considered BVP), V is the space of test functions, l is a linear operator dependent on v.

ICCS Camera Ready Version 2023
To cite this paper please use the final published version:
DOI: 10.1007/978-3-031-36027-5_41

- *minimization of the potential energy functional*

$$\min_u I(u), \quad I(u) = 1/2\, b(u,u) - l(u)\ . \tag{6}$$

In both cases, corresponding boundary conditions have to be satisfied.

In the variational formulations one may deal with the Petrov-Galerkin approach when u and v are different functions from each other, and the Bubnov-Galerkin one, when u and v are the same. Assuming a trial function u locally defined on each subdomain Ω_i within the domain Ω one may obtain a global-local formulation of the Petrov-Galerkin type (MLPG). The test function v may be defined here in various ways. In particular, one may assume it as also given locally in each subdomain Ω_i. Usually, the test function is assumed to be equal to zero elsewhere, though it may be defined in many other ways.

In *the MLPG5 formulation* [17] the Heaviside type test function is assumed. In each subdomain Ω_i around a node P_i, $i = 1, 2, ..., N$, in the given domain Ω (e.g. in each Voronoi polygon in 2D) the test function is equal to one ($v = 1$) in Ω_i and is assumed to be zero outside. Hence any derivative of v is also equal to zero in the whole domain Ω. Therefore, relevant expressions in the functional (5) $b(u, v)$ and in $l(v)$ vanish, reducing in this way the amount of calculations involved.

Let us consider the following two dimensional elliptic problem

$$\nabla^2 u = f(P), \quad P(x,y) \in \Omega, \quad u\big|_\Gamma = 0\ , \tag{7}$$

which satisfies the differential equation of the second order with Dirichlet conditions on the boundary Γ of the domain Ω.

When the variational form is derived directly from the (6) by integration over the domain, *the first nonsymmetric variational form* is considered:

$$\int_\Omega \left(u_{xx} + u_{yy}\right) v\, d\Omega = \int_\Omega f\, v\, d\Omega\ , \qquad u \in H^2, \quad v \in H^0\ . \tag{8}$$

After differentiation by parts, *the symmetric Galerkin form* is obtained

$$\int_\Omega \left(u_x v_x + u_y v_y\right) d\Omega + \int_\Omega f\, v\, d\Omega = \int_\Gamma \left(u_x n_x + u_y n_y\right) v\, d\Gamma\ , \quad u \in H^1, \quad v \in H^1\ , \tag{9}$$

where n_x and n_y denotes the normal vectors. For the Heaviside test function, *the MLPG5 formulation* is as follows*:*

$$\int_{\Omega_i} f\, v\, d\Omega_i = \int_{\Gamma_i} \left(u_x n_x + u_y n_y\right) v\, d\Gamma_i\ , \quad u \in H^1, \quad v \in H^1\ . \tag{10}$$

All variants of the global (weak) formulation of the multipoint method may be realized by the meshless MFDM using regular or totally irregular meshes like it is in the case of the local formulation of BVP.

In all weak formulations of the multipoint MFDM developed here, the unknown trial function u and its derivatives are always approximated using the multipoint finite difference formulas (3) or (4) (MWLS technique based on the stencil subdomain). However, assumption and discretization of the test function v and its deriva-

ICCS Camera Ready Version 2023
To cite this paper please use the final published version:
DOI: 10.1007/978-3-031-36027-5_41

tives at Gauss points may be done in many ways – calculated by any type of approximation as well as by the simple interpolation on the integration subdomain, which can be other than the stencil subdomain, e.g. Delaunay triangle or Voronoi polygon. This is a direct result of the assumption that u and v functions may be different, and a choice of the test function v should not influence the BVP solution. This statement was positively tested using benchmark problems (Fig. 2).

4 Numerical analysis

4.1 Benchmark tests

Several benchmark tests of the application of the multipoint method to the BVP globally formulated were carried out. The multipoint approach was tested in the Galerkin as well as in the MLPG5 formulations, and minimization of the potential energy functional. The solutions were compared with the corresponding ones obtained for the strong (general and specific) multipoint MFDM formulations. The results of the numerical tests of 2D Poisson's problem done are presented in Figs.1-3 (*h* is the distance between the nodes).

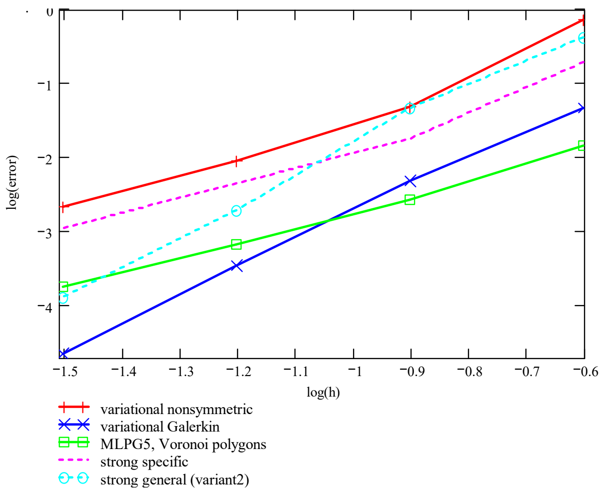

Fig. 1. Convergence of the strong formulations (Multipoint, 3rd approx. order) and the various weak (nonsymmetric, Galerkin, MLPG5) ones

The type of approximation of the test function does not significantly influence the results (Fig. 2). In the benchmark tests (Fig. 3) the quadrilateral integration subdomains (Voronoi polygons, integration around nodes) give slightly better results than the triangular ones (Delaunay triangles, integration between nodes). The clear advantage of the MLPG5 formulation relies on a significant reduction of numerical operations needed to obtain the final solution of the problems tested, and, as a consequence – the computational time reduction.

ICCS Camera Ready Version 2023
To cite this paper please use the final published version:
DOI: 10.1007/978-3-031-36027-5_41

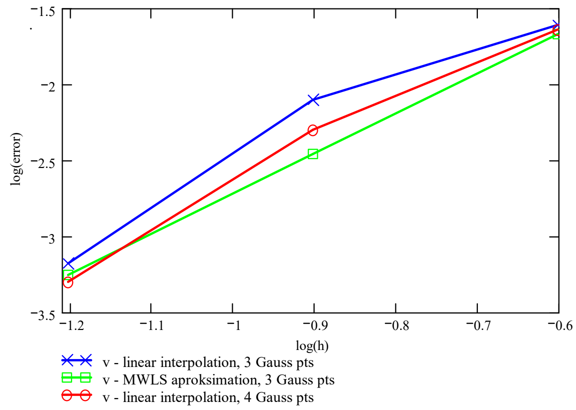

Fig. 2. Influence of the test function interpolation, nonsymmetric variational form

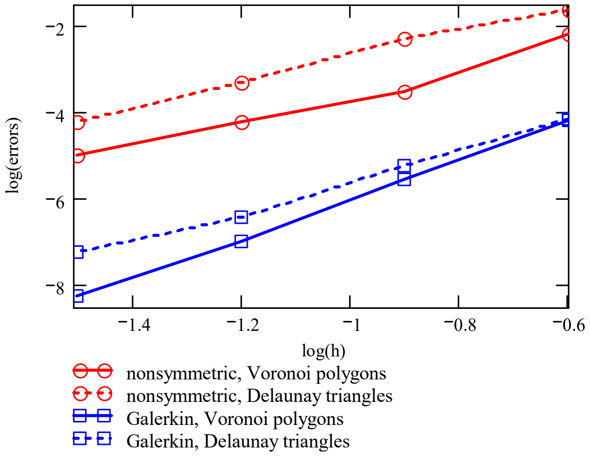

Fig. 3. Influence of the integration around (Voronoi polygons) or between nodes (Delaunay triangles)

4.2 Weak and strong formulations in nonlinear and multiscale analyses

The problem formulation may have an influence on the algorithm and the results, especially in the case of more demanding problems, such as multiscale analysis of heterogeneous materials and elastic-plastic problems [18]. The multiscale analysis was carried out by the multipoint method using weak formulations. On the other side, in the elastic-plastic analysis of physically nonlinear problems – the multipoint approach was applied to the strong formulation. In both types of analyses, the problems deal with the jump between types or states of materials.

The oscillations occurred on the interface of the matrix and inclusions of the heterogeneous materials when the nodes of one stencil belong to different material types (Fig. 4). In this situation the only solution was to adjust the stencil to the inclusion

ICCS Camera Ready Version 2023
To cite this paper please use the final published version:
DOI: 10.1007/978-3-031-36027-5_41

distribution [19]. In nonlinear analysis, the similar situation was expected at the elastic-plastic boundary. However, the preliminary numerical results [20] do not demonstrate the oscillations phenomenon (Fig. 5). It seems, that the difference is caused by the assumed type of the problem formulation. Further research is planned.

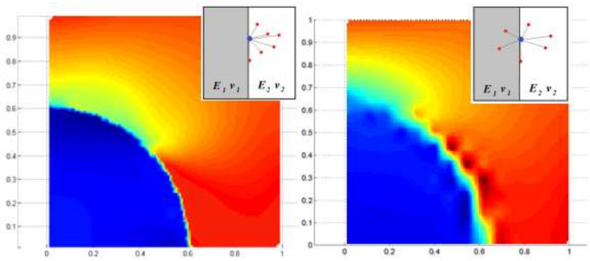

Fig. 4. RVE solution (strain ε_{xx}) obtained on the stencil generated either independently on the inclusion distribution or adjusted to it

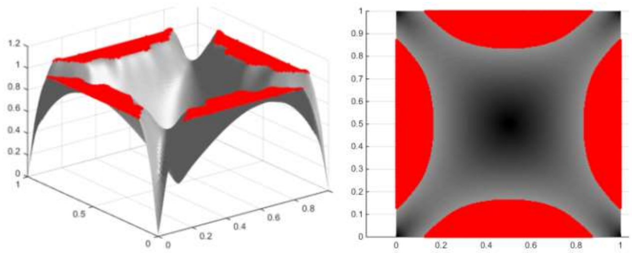

Fig. 5. The total shear stress, coarse and fine mesh (red color depict the plastic zones)

5 Final remarks

The paper presents an extension of the novel high order multipoint MFDM to the various weak (global) formulations of the boundary value problems. Taken into account were some versions of the weak formulation including the variational Galerkin, local Petrov-Galerkin (MLPG), and minimum of the total potential energy.

The various tests carried out confirm that the multipoint approach may be a useful solution tool for the analysis of boundary value problems given in the global formulation and provide valuable results, close enough to the local (strong) form. Additionally, the advantages of the MLPG5 approach due the Heaviside type test function used – reducing the amount of calculations involved – may be noticed.

The specific or general multipoint MFD operators should be applied to the trial function in the case of the global or global-local formulations of the analyzed problem, and numerical integration is additionally required here. The type of approximation of the test function is not very important here.

Although the results obtained by the multipoint technique are close enough in both strong and weak formulations in general, in the case of more demanding problems, such as nonlinear or multiscale analyses, the problem formulation may influence on the computational algorithm of the method. In the case of the weak formulation, it

ICCS Camera Ready Version 2023
To cite this paper please use the final published version:
DOI: 10.1007/978-3-031-36027-5_41

may be necessary to match the stencils (MFD stars) to the interfaces of the different types or states of the material. Further research is planned.

References

1. Lancaster, P. and Salkauskas, K., Surfaces generated by moving least-squares method, Math Comp 155(37):141-158 (1981).
2. Liszka, T., Orkisz, J., The finite difference method at arbitrary irregular grids and its applications in applied mechanics, Comput Struct, 11, 83–95 (1980).
3. Nayroles, B., Touzot, G., and Villon, P., Generalizing the finite element method: Diffuse approximation and diffuse elements, Comput.Mech., 10(5), 307–318 (1992).
4. Belytschko, T., Lu, Y. Y., and Gu, L. , Element-free Galerkin methods, Int. J. Numer. Methods Eng., 37(2), 229–256 (1994).
5. Onate, E., Idelsohn, S. R., Zienkiewicz, O. C., and Taylor, R. L., A finite point method in computational mechanics: Applications to convective transport and fluid flow, Int. J. Numer. Methods Eng., 39(22), 3839–3866 (1996).
6. Atluri, S.N. and Zhu, T. , A new meshless local Petrov–Galerkin (MLPG) approach in computational mechanics. Comput. Mech., 22: 117–127 (1998).
7. Lucy, L.B. , A numerical approach to the testing of the fission hypothesis, Astron. J. 82: 1013–1024 (1977).
8. Gingold, R.A. and Monaghan, J.J., Smoothed particle hydrodynamics: theory and application to non-spherical stars. Monthly Notices R. Astron.Soc. 181: 375–389 (1977).
9. Liu, W.K., Jun, S. and Zhang, Y.F., Reproducing kernel particle methods, Int. J. Numer. Methods Eng. 20: 1081–1106 (1995).
10. Duarte, C.A. and Oden, J.T. , An hp adaptive method using clouds. Comput. Methods Appl. Mech. Eng. 139: 237–262 (1996).
11. Melenk, J.M. and Babuska I., The partition of unity finite element method: basic theory and applications. Comput Methods Appl Mech Eng 139:289–314 (1996).
12. Orkisz, J., Finite Difference Method (Part III). In:M.Kleiber (ed.) Handbook of Computational Solid Mechanics, Springer-Verlag, Berlin, pp.336-432 (1998).
13. Jaworska, I.: On the ill-conditioning in the new higher order multipoint method, Comput. Math. Appl. 66(3), 238-249 (2013).
14. Jaworska, I., Orkisz, J.: Higher order multipoint method – from Collatz to meshless FDM, Eng. Anal. Bound. Elem. 50, 341–351 (2015).
15. Milewski, S., Meshless Finite Difference Method with Higher Order Approximation – Applications in Mechanics, Arch. of Comp. Meth in Eng, 19(1): 1–49 (2012).
16. Collatz, L., Numerische Behandlung von Differential-gleichungen, Springer-Verlag, Berlin-Heidelberg (1955).
17. Atluri S.N., Shen S., The Meshless Local Petrov-Galerkin (MLPG) Method, Tech Science Press (2002).
18. Jaworska, I.: Multipoint Meshless FD Schemes Applied to Nonlinear and Multiscale Analysis, Computational Science, 13353, ICCS 2022, 55-68 (2022).
19. Jaworska, I.: On some aspects of the Meshless FDM application for the heterogeneous materials, Int. J. Multiscale Comput. Eng. 15(4), 359–378 (2017).
20. Jaworska, I.: Generalization of the Multipoint meshless FDM application to the nonlinear analysis, Comput. Math. Appl. 87(3), 1-11 (2021).
21. Chen J-S., Hillman M., Chi S-W.: Meshfree methods: progress made after 20 years. J Eng Mech 143(4):04017001 (2017).
22. Nguyen, V.P., Rabczuk, T., Bordas, S., Duflot, M.: Meshless methods: a review and computer implementation aspects, Math Comput Simul, 79, 763–813 (2008).

ICCS Camera Ready Version 2023
To cite this paper please use the final published version:
DOI: 10.1007/978-3-031-36027-5_41