Biharmonic scattered data interpolation based on the Method of Fundamental Solutions

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Abstract. The two-dimensional scattered data interpolation problem is investigated. In contrast to the traditional Method of Radial Basis Functions, the interpolation problem is converted to a higher order (biharmonic or modified bi-Helmholtz) partial differential equation supplied with usual boundary conditions as well as pointwise interpolation conditions. To solve this fourth-order problem, the Method of Fundamental Solutions is used. The source points, which are needed in the method, are located partly in the exterior of the domain of the corresponding partial differential equation and partly in the interpolation points. This results in a linear system with possibly large and fully populated matrix. To make the computations more efficient, a localization technique is applied, which splits the original problem into a sequence of local problems. The system of local equations is solved in an iterative way, which mimics the classical overlapping Schwarz method. Thus, the problem of large and ill-conditioned matrices is completely avoided. The method is illustrated via a numerical example.

Keywords: Scattered data interpolation · Method of Fundamental Solutions · Localization.

1 Introduction

The scattered data interpolation problem is a relatively new mathematical problem which goes back to the pioneering work of Shepard [11]. His method was based on weighted averages, the weights of which are inversely proportional to some powers of the distances between the interpolation points and the point in which the interpolation function is to be evaluated. Later, a much more powerful family of methods was developed, the method of Radial Basis Functions (RBFs), see e.g. [6], [2]. Here the interpolation function is sought in the following form:

$$u(x) := \sum_{j=1}^{N} \alpha_j \cdot \Phi(x - x_j), \tag{1}$$

where Φ is a predefined radial (i.e., circularly symmetric) function, $x_1, x_2, ..., x_N$ are predefined interpolation points scattered in the plane $\mathbf{R}^2$ without having any grid or mesh structure, $x \in \mathbf{R}^2$ is an evaluation point. The method can be

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defined for higher dimensional interpolation problems in a similar way: here we restrict ourselves to 2D problems.

The a priori unknown coefficients $\alpha_1, \alpha_2, ..., \alpha_N$ can be determined by enforcing the *interpolation conditions*:

$$\sum_{j=1}^{N} \alpha_j \cdot \Phi(x_k - x_j) = u_k, \qquad k = 1, 2, ..., N, \tag{2}$$

where the predefined values $u_1, u_2, ..., u_N$ are associated to the interpolation points $x_1, x_2, ..., x_N$.

For the RBF Φ, several choices have been proposed. Some popular techniques are as follows (written in polar coordinates, for the sake of simplicity):

- Multiquadrics: $\Phi(r) := \sqrt{c^2 + r^2}$;
- Inverse multiquadrics: $\Phi(r) := \frac{1}{\sqrt{c^2+r^2}}$;
- Thin plate splines: $\Phi(r) := r^2 \cdot \log(r)$;
- Polyharmonic splines: $\Phi(r) := r^{2k} \cdot \log(r)$ (where k is a predefined positive integer);
- Gauss functions: $\Phi(r) := e^{-c^2 \cdot r^2}$;

and so forth (in the above formulations, c denotes a predefined scaling constant). The above radial basis functions are *globally supported*, therefore the matrix of the system (2) is fully populated and sometimes severely ill-conditioned, which may cause computational difficulties, especially when the number of the interpolation points is large.

To overcome this difficulty, several methods have been developed. One of them is the use of *compactly supported* radial basis functions (Wendland functions, see e.g. [13]). Thus, the matrix of the system (2) becomes sparse, which is advantageous from computational point of view.

Another technique is a generalization of the concept of the thin plate splines. Utilizing the fact that the radial basis function of the thin plate spline $\Phi(r) = r^2 \cdot \log(r)$ is biharmonic (except for the origin), the interpolation problem can be converted to a problem defined for the biharmonic equation supplied with some usual boundary conditions along the boundary and also with the interpolation conditions at the interpolation points. Thus, instead of a scattered data interpolation problem, a fourth-order partial differential equation is to be solved (supplied with some unusual conditions, i.e. pointwise interpolation conditions).

In its original form, the resulting biharmonic problem is solved by a finite volume method. The cell system which the finite volume method is performed on, is preferably defined by a *quadtree subdivision* algorithm, which automatically generates local refinements in the vicinity of the interpolation points. The solution procedure can be embedded in a natural multi-level context. This makes the method quite economic from computational point of view. Note that the above biharmonic equation can be replaced with more general fourth-order partial differential equations e.g. the modified Helmholtz equation. See [3] for details. However, the method can be considered a 'quasi-meshfree' method only, though

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the construction of cell system as well as the solution process are completely controlled by the interpolation points.

In this paper, the above outlined strategy based on the solution of a biharmonic equation is connected with a truly *meshless* method, namely, the Method of Fundamental Solutions (MFS), see [5]. This method results in a linear system of equations with a fully populated (but often ill-conditioned) matrix. The approach has been applied to biharmonic equations as well, see [10]. The computational difficulties can be reduced by introducing *localization* techniques which convert the original problem to a set of smaller problems. See e.g. [1], [12].

In the following, the MFS-based solution technique is generalized to the biharmonic interpolation problem using a special localization method which is based on the traditional Schwarz alternating method. This results in a special iterative method and splits the original problem into several local (and much less) subproblems. The method is illustrated through a simple example.

2 Biharmonic interpolation

The main idea of this type of interpolation is to convert original interpolation problem to a higher order partial differential equation supplied with the interpolation conditions as special pointwise boundary conditions. For second-order partial differential equations, this results in an ill-posed problem, but for fourth-order equations, this does not remain the case.

Let $\Omega \subset \mathbf{R}^2$ be a two-dimensional, bounded and sufficiently smooth domain, and let $x_1, x_2, ..., x_N \in \Omega$ be predefined interpolation points. Denote by $u_1, u_2, ..., u_N \in \mathbf{R}$ the values associated to the interpolation points. The biharmonic interpolation function u is expected to satisfy the biharmonic equation in the domain Ω except for the interpolation points:

$$\Delta\Delta u = 0, \qquad \text{in } \Omega \setminus \{x_1, x_2, ..., x_N\}, \tag{3}$$

where Δ denotes the Laplace operator. Along the boundary $\Gamma := \partial\Omega$, some usual boundary condition can be prescribed, e.g. Dirichlet boundary condition:

$$u|_\Gamma = u_0, \qquad \frac{\partial u}{\partial n}|_\Gamma = v_0, \tag{4}$$

or Navier boundary condition:

$$u|_\Gamma = u_0, \qquad \Delta u|_\Gamma = w_0, \tag{5}$$

where u_0, v_0, w_0 are predefined, sufficiently regular boundary functions. At the interpolation points, the interpolation conditions

$$u(x_k) = u_k, \qquad k = 1, 2, ..., N, \tag{6}$$

are prescribed.

It is known that in spite of the pointwise defined interpolation conditions, the problem (3)–(6) has a unique solution in a closed subspace of the Sobolev

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space $H^2(\Omega)$. See [3] for details. Note that the interpolation conditions defined in discrete points do not make the problem ill-posed due to the fact that the Dirac functionals $u \to u(x_k)$ are continuous in the Sobolev space $H^2(\Omega)$ (but not in $H^1(\Omega)$, which is the usual basis of second-order elliptic boundary value problems). It should be pointed out that the biharmonic equation (3) can be replaced with other fourth-order differential equations e.g. the modified bi-Helmholtz equation:

$$(\Delta - c^2 I)^2 u = 0, \qquad \text{in } \Omega \setminus \{x_1, x_2, ..., x_N\}, \tag{7}$$

where I denotes the identity operator and c is a predefined constant which plays some scaling role.

The above idea converts the original *interpolation problem* into the solution of a (fourth-order) *partial differential equation* which seems to be much more difficult from computational point of view. However, if this partial differential equation is solved by a computationally efficient method, e.g. on a non-uniform, non-equidistant cell system using finite volume schemes and multi-level techniques, the necessary computational cost can significantly be reduced. Such a non-equidistant, non-uniform cell system can be created by the help of the well-known *quadtree algorithm* controlled by the interpolation points. The algorithm results in local refinements in the vicinity of the interpolation points and makes it possible to build up a multi-level solution technique in a natural way. For details, see [3], [4] . Nevertheless, the accuracy of the above finite volume schemes is moderate. Moreover, the evaluation points of the interpolation function are fixed to be the cell centers. In this paper, another solution technique is presented, which is based on the Method of Fundamental Solutions.

3 The Method of Fundamental Solutions applied to the biharmonic interpolation problem

First, let us briefly recall the main concepts and ideas of the Method of Fundamental Solutions. For details, see e.g. [5].

The Method of Fundamental Solutions (MFS) is now a quite popular method for solving partial differential equations. It is truly meshless, i.e. it requires neither domain nor boundary grid or mesh structure. If the differential equation has the form

$$Lu = 0, \qquad \text{in } \Omega,$$

where L is a linear partial differential operator, and Φ denotes a fundamental solution of the operator L, then the MFS produces the approximate solution in the following form:

$$u(x) = \sum_{j=1}^{M} \alpha_j \cdot \Phi(x - s_j),$$

where $s_1, s_2, ..., s_M$ are predefined *source points* in the exterior of Ω. The a priori unknown coefficients $\alpha_1, \alpha_2, ..., \alpha_M$ can be computed by enforcing the boundary conditions in some predefined $x_1, x_2, ..., x_N$ *boundary collocation points*. In the

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simplest case, when L is of second order and Dirichlet boundary condition is prescribed, this results in the following linear system of algebraic equations:

$$\sum_{j=1}^{M} \alpha_j \cdot \Phi(x_k - s_j) = u(x_k), \qquad k = 1, 2, ..., N.$$

Recently, the approach has been generalized also to inhomogeneous problems, see e.g. [14], and also for more general equations, see [8].

In the case of the biharmonic equation

$$\Delta\Delta u = 0, \tag{8}$$

the MFS defines the approximate solution in the following form [10]:

$$u(x) := \sum_{j=1}^{M} \alpha_j \cdot \Phi(x - s_j) + \sum_{j=1}^{M} \beta_j \cdot \Psi(x - s_j). \tag{9}$$

Here $s_1, s_2, ..., s_M$ are again exterior source points. Φ denotes the following harmonic fundamental solution:

$$\Phi(x) := \frac{1}{2\pi} \log ||x||,$$

and Ψ denotes the following biharmonic fundamental solution:

$$\Psi(x) := \frac{1}{8\pi} ||x||^2 \log ||x|| - \frac{1}{8\pi} ||x||^2$$

(the symbol $|| \cdot ||$ denotes the usual Euclidean norm in $\mathbf{R^2}$).

The above definitions imply that $\Delta\Psi = \Phi$. This will simplify the later calculations.

Suppose, for simplicity, that the biharmonic equation (8) is supplied with Navier boundary condition:

$$u|_\Gamma = u_0, \qquad \Delta u|_\Gamma = w_0. \tag{10}$$

Then the coefficients $\alpha_1, ..., \alpha_M$, $\beta_1, ..., \beta_M$ can be calculated by enforcing the boundary conditions. Utilizing the equality $\Delta\Psi = \Phi$, this results in the following linear system of equations:

$$\begin{aligned} \sum_{j=1}^{M} \alpha_j \cdot \Phi(x_k - s_j) + \sum_{j=1}^{M} \beta_j \cdot \Psi(x_k - s_j) = u_0(x_k), \quad k = 1, 2, ..., N, \\ \sum_{j=1}^{M} \beta_j \cdot \Phi(x_k - s_j) = w_0(x_k), \quad k = 1, 2, ..., N, \end{aligned} \tag{11}$$

where $x_1, x_2, ..., x_N \in \Gamma$ are predefined boundary collocation points. In general, the numbers N and M need not be equal. If they differ, then the system

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(11) should be solved e.g. in the sense of least squares. (11) is often overdetermined, i.e., the number of collocation points is (much) greater than the number of sources. Even if $N = M$, the above system of equations may be severely ill-conditioned, especially when the sources are located far from the boundary of the domain Ω. Moreover, the matrix of the system is fully populated. Therefore, the practical implementation of the method may often be difficult from computational point of view.

3.1 Solution of the biharmonic problem by overlapping Schwarz method

To circumvent the above mentioned computational problems, the traditional Schwarz overlapping method [7] is applied. Such a method splits the original problem into several smaller ones. It was originally defined for second-order partial differential equations, however, the idea can easily be generalized for our biharmonic problem. For the sake of simplicity, assume that Ω is a rectangle, $\Omega = \Omega_1 \cup \Omega_2$, where $\Omega_1 = (-\pi + h, h) \times (0, \pi)$, $\Omega_2 = (-h, \pi - h) \times (0, \pi)$. The predefined value $0 < h < \pi/2$ characterizes the overlap, see Fig. (1). This situation is a *strong overlap*, i.e. the distance of Γ_1 and Γ_2 is positive ($= 2h$), where $\Gamma_1 := \partial\Omega_1 \cap \Omega_2$, and $\Gamma_2 := \partial\Omega_2 \cap \Omega_1$.

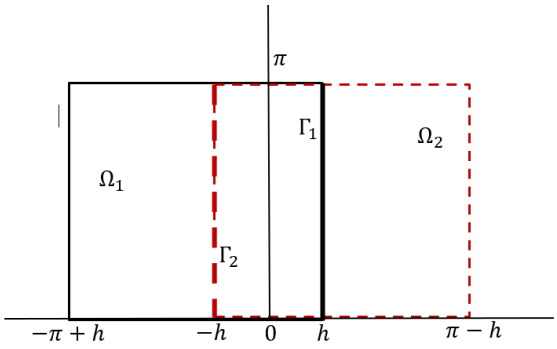

Fig. 1. Strongly overlapping subdomains. The distance of Γ_1 and Γ_2 is positive.

First, consider the pure biharmonic problem supplied with Navier boundary conditions but without interpolation conditions:

$$\begin{aligned} \Delta\Delta u &= 0, \qquad \text{in } \Omega, \\ u|_\Gamma &= u_0, \qquad \Delta u|_\Gamma = w_0. \end{aligned} \tag{12}$$

The Schwarz overlapping method approximates the solution with the following sequence of functions. Starting from a function u_1, for which $u_1 \in H^2(\Omega)$, $\Delta u_1 \in$

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$H^1(\Omega)$, define the sequence of the following subproblems for $n = 1, 2, \ldots$:

$$\begin{gathered} \Delta\Delta u_{n+1/2} = 0, \qquad \text{in } \Omega_1, \\ u_{n+1/2}|_{\partial\Omega_1\setminus\Gamma_1} = u_0|_{\partial\Omega_1\setminus\Gamma_1}, \qquad \Delta u_{n+1/2}|_{\partial\Omega_1\setminus\Gamma_1} = w_0|_{\partial\Omega_1\setminus\Gamma_1}, \\ u_{n+1/2}|_{\Gamma_1} = u_n|_{\Gamma_1}, \qquad \Delta u_{n+1/2}|_{\Gamma_1} = \Delta u_n|_{\Gamma_1}, \end{gathered} \tag{13}$$

$$\begin{gathered} \Delta\Delta u_{n+1} = 0, \qquad \text{in } \Omega_2, \\ u_{n+1}|_{\partial\Omega_2\setminus\Gamma_2} = u_0|_{\partial\Omega_2\setminus\Gamma_2}, \qquad \Delta u_{n+1}|_{\partial\Omega_2\setminus\Gamma_2} = w_0|_{\partial\Omega_2\setminus\Gamma_2}, \\ u_{n+1}|_{\Gamma_2} = u_{n+1/2}|_{\Gamma_2}, \qquad \Delta u_{n+1}|_{\Gamma_2} = \Delta u_{n+1/2}|_{\Gamma_2}. \end{gathered} \tag{14}$$

Now we will prove that the above defined Schwarz sequence converges to the exact solution u^*. Denote by e_n, $e_{n+1/2}$ the corresponding errors: $e_n := u_n - u^*$ (in Ω_2) and $e_{n+1/2} := u_{n+1/2} - u^*$ (in Ω_1). Then, obviously, the error functions satisfy the following biharmonic problems:

$$\begin{gathered} \Delta\Delta e_{n+1/2} = 0, \qquad \text{in } \Omega_1, \\ e_{n+1/2}|_{\partial\Omega_1\setminus\Gamma_1} = 0, \qquad \Delta e_{n+1/2}|_{\partial\Omega_1\setminus\Gamma_1} = 0, \\ e_{n+1/2}|_{\Gamma_1} = e_n|_{\Gamma_1}, \qquad \Delta e_{n+1/2}|_{\Gamma_1} = \Delta e_n|_{\Gamma_1}, \end{gathered} \tag{15}$$

$$\begin{gathered} \Delta\Delta e_{n+1} = 0, \qquad \text{in } \Omega_2, \\ e_{n+1}|_{\partial\Omega_2\setminus\Gamma_2} = 0, \qquad \Delta e_{n+1}|_{\partial\Omega_2\setminus\Gamma_2} = 0, \\ e_{n+1}|_{\Gamma_2} = e_{n+1/2}|_{\Gamma_2}, \qquad \Delta e_{n+1}|_{\Gamma_2} = \Delta e_{n+1/2}|_{\Gamma_2}. \end{gathered} \tag{16}$$

Now express $e_n|_{\Gamma_1}$ and $\Delta e_n|_{\Gamma_1}$ in terms of trigonometric Fourier series (with respect to the variable y). Due to the boundary conditions along the horizontal sides, it is sufficient to use *sinusoidal* Fourier series. Using the traditional notations x, y for the spatial variables, we have:

$$e_n|_{\Gamma_1} = e_n(h, y) = \sum_{k=1}^{\infty} \alpha_k^{(n)} \sin ky, \quad \Delta e_n|_{\Gamma_1} = \Delta e_n(h, y) = \sum_{k=1}^{\infty} \beta_k^{(n)} \sin ky. \tag{17}$$

Straightforward calculations show that, in Ω_1, the solution $e_{n+1/2}$ has the form:

$$\begin{aligned} e_{n+1/2}(x, y) = \sum_{k=1}^{\infty} \Big(& A_k^{(n+1/2)} \sinh k(x + \pi - h) + \\ & + B_k^{(n+1/2)} (x + \pi - h) \cosh k(x + \pi - h) \Big) \sin ky \end{aligned} \tag{18}$$

with some coefficients $A_k^{(n+1/2)}$, $B_k^{(n+1/2)}$. Indeed, one can easily check that each term in the Fourier series (as a function of x and y) is biharmonic. Moreover:

$$\Delta e_{n+1/2}(x, y) = \sum_{k=1}^{\infty} 2k B_k^{(n+1/2)} \sinh k(x + \pi - h) \sin ky, \tag{19}$$

therefore both $e_{n+1/2}$ and $\Delta e_{n+1/2}$ vanish along $\partial\Omega_1 \setminus \Gamma_1$.

The equalities

$$e_{n+1/2}|_{\Gamma_1} = e_n|_{\Gamma_1}, \qquad \Delta e_{n+1/2}|_{\Gamma_1} = \Delta e_n|_{\Gamma_1}$$

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imply that the vectors of coefficients $\begin{pmatrix} A_k^{(n+1/2)} \\ B_k^{(n+1/2)} \end{pmatrix}$ satisfy the systems of equations:

$$\begin{pmatrix} \sinh k\pi & \pi\cosh k\pi \\ 0 & 2k\sinh k\pi \end{pmatrix} \begin{pmatrix} A_k^{(n+1/2)} \\ B_k^{(n+1/2)} \end{pmatrix} = \begin{pmatrix} \alpha_k^{(n)} \\ \beta_k^{(n)} \end{pmatrix}.$$

Now the traces of $e_{n+1/2}$ and $\Delta e_{n+1/2}$ along Γ_2 can be calculated without difficulty:

$$\begin{aligned} e_{n+1/2}|_{\Gamma_2} = e_{n+1/2}(-h,y) = \sum_{k=1}^{\infty} \Big(A_k^{(n+1/2)} \sinh k(\pi-2h) + \\ + B_k^{(n+1/2)} (\pi-2h)\cosh k(\pi-2h) \Big) \sin ky =: \\ =: \sum_{k=1}^{\infty} \alpha_k^{(n+1/2)} \sin ky, \end{aligned} \tag{20}$$

$$\begin{aligned} \Delta e_{n+1/2}|_{\Gamma_2} = \Delta e_{n+1/2}(-h,y) = \sum_{k=1}^{\infty} 2k B_k^{(n+1/2)} \sinh k(\pi-2h) \sin ky =: \\ =: \sum_{k=1}^{\infty} \beta_k^{(n+1/2)} \sin ky. \end{aligned} \tag{21}$$

That is, the vectors of the Fourier coefficients $\begin{pmatrix} \alpha_k^{(n+1/2)} \\ \beta_k^{(n+1/2)} \end{pmatrix}$ can be expressed with the help of $\begin{pmatrix} \alpha_k^{(n)} \\ \beta_k^{(n)} \end{pmatrix}$ as follows:

$$\begin{pmatrix} \alpha_k^{(n+1/2)} \\ \beta_k^{(n+1/2)} \end{pmatrix} = \begin{pmatrix} \sinh k(\pi-2h) & (\pi-2h)\cosh k(\pi-2h) \\ 0 & 2k\sinh k(\pi-2h) \end{pmatrix} \begin{pmatrix} A_k^{(n+1/2)} \\ B_k^{(n+1/2)} \end{pmatrix} =$$

$$= \begin{pmatrix} \sinh k(\pi-2h) & (\pi-2h)\cosh k(\pi-2h) \\ 0 & 2k\sinh k(\pi-2h) \end{pmatrix} \begin{pmatrix} \sinh k\pi & \pi\cosh k\pi \\ 0 & 2k\sinh k\pi \end{pmatrix}^{-1} \begin{pmatrix} \alpha_k^{(n)} \\ \beta_k^{(n)} \end{pmatrix}.$$

In a quite similar way, the error function e_{n+1} can be expressed in Ω_2 as:

$$\begin{aligned} e_{n+1}(x,y) = \sum_{k=1}^{\infty} \Big(A_k^{(n+1)} \sinh k(x-\pi+h) + \\ + B_k^{(n+1)} (x-\pi+h)\cosh k(x-\pi+h) \Big) \sin ky \end{aligned} \tag{22}$$

with some coefficients $A_k^{(n+1)}$, $B_k^{(n+1)}$. Moreover:

$$\Delta e_{n+1}(x,y) = \sum_{k=1}^{\infty} 2k B_k^{(n+1)} \sinh k(x-\pi+h) \sin ky, \tag{23}$$

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where the vectors of coefficients $\begin{pmatrix} A_k^{(n+1)} \\ B_k^{(n+1)} \end{pmatrix}$ satisfy the systems of equations:

$$\begin{pmatrix} \sinh k(-\pi) & -\pi\cosh k(-\pi) \\ 0 & 2k\sinh k(-\pi) \end{pmatrix} \begin{pmatrix} A_k^{(n+1)} \\ B_k^{(n+1)} \end{pmatrix} = \begin{pmatrix} \alpha_k^{(n+1/2)} \\ \beta_k^{(n+1/2)} \end{pmatrix}.$$

Calculating the traces of e_{n+1} and Δe_{n+1} along Γ_1, we obtain:

$$\begin{aligned} e_{n+1}|_{\Gamma_1} = e_{n+1}(h,y) = \sum_{k=1}^{\infty} \Big(A_k^{(n+1)} \sinh k(2h-\pi) + \\ + B_k^{(n+1)} (2h-\pi)\cosh k(2h-\pi) \Big) \sin ky =: \\ =: \sum_{k=1}^{\infty} \alpha_k^{(n+1)} \sin ky, \end{aligned} \tag{24}$$

$$\begin{aligned} \Delta e_{n+1}|_{\Gamma_1} = \Delta e_{n+1}(h,y) = \sum_{k=1}^{\infty} 2k B_k^{(n+1)} \sinh k(2h-\pi) \sin ky =: \\ =: \sum_{k=1}^{\infty} \beta_k^{(n+1)} \sin ky. \end{aligned} \tag{25}$$

Consequently, the vectors of the Fourier coefficients $\begin{pmatrix} \alpha_k^{(n+1)} \\ \beta_k^{(n+1)} \end{pmatrix}$ can be expressed with the help of $\begin{pmatrix} \alpha_k^{(n+1/2)} \\ \beta_k^{(n+1/2)} \end{pmatrix}$ exactly in the same way than $\begin{pmatrix} \alpha_k^{(n+1/2)} \\ \beta_k^{(n+1/2)} \end{pmatrix}$ with the help of $\begin{pmatrix} \alpha_k^{(n)} \\ \beta_k^{(n)} \end{pmatrix}$. This implies that after a complete Schwarz iteration:

$$\begin{pmatrix} \alpha_k^{(n+1)} \\ \beta_k^{(n+1)} \end{pmatrix} = M_k^2 \begin{pmatrix} \alpha_k^{(n)} \\ \beta_k^{(n)} \end{pmatrix},$$

where

$$M_k = \begin{pmatrix} \sinh k(\pi-2h) & (\pi-2h)\cosh k(\pi-2h) \\ 0 & 2k\sinh k(\pi-2h) \end{pmatrix} \begin{pmatrix} \sinh k\pi & \pi\cosh k\pi \\ 0 & 2k\sinh k\pi \end{pmatrix}^{-1}.$$

Standard calculations show that

$$\begin{pmatrix} \sinh k\pi & \pi\cosh k\pi \\ 0 & 2k\sinh k\pi \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{\sinh k\pi} & -\frac{\pi\cosh k\pi}{2k(\sinh k\pi)^2} \\ 0 & \frac{1}{2k\sinh k\pi} \end{pmatrix}.$$

Consequently, M_k is an upper triangular matrix, and both diagonal entries are equal to $\frac{\sinh k(\pi-2h)}{\sinh k\pi}$. This implies that both eigenvalues of M_k^2 are equal to $\left(\frac{\sinh k(\pi-2h)}{\sinh k\pi}\right)^2$. And since $0 < h < \frac{\pi}{2}$, these eigenvalues are less than 1, i.e., the Schwarz iteration is convergent. Theorem is proven.

The above theorem can be generalized to the case when the number of subdomains is greater that 2. Details are omitted.

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3.2 Localization of the MFS for the biharmonic equation based on overlapping Schwarz method

In practice, the idea of the overlapping Schwarz method can be used for creating a special localization technique. Here we briefly outline this method applied to the biharmonic equation. For the localization of the biharmonic equation, without Schwarz method, see e.g. [1], [12], [9].

Consider a point set $S_0 := \{x_1, x_2, ..., x_N\}$ scattered in the domain Ω. Denote by S_b the set of predefined boundary collocation points: $S_b := \{\hat{x}_1, \hat{x}_2, ..., \hat{x}_M\}$, and define the set $S := S_0 \cup S_b$.

For a given central point $x_0^{(i)} \in S_0$, define the circle centered at $x_0^{(i)}$ with the prescribed radius $R^{(i)}$ (the *local subdomains*):

$$\Omega^{(i)} := \{x \in \mathbf{R}^2 : \ ||x - x_0^{(i)}|| < R^{(i)}\}$$

Define the sets:

$$S_0^{(i)} := S_0 \cap \Omega^{(i)} = \{x_1^{(i)}, ..., x_{N_p^{(i)}}^{(i)}\}, \quad S_b^{(i)} := S_b \cap \Omega^{(i)} = \{\hat{x}_1^{(i)}, ..., \hat{x}_{N_b^{(i)}}^{(i)}\}.$$

Note that the set $S_b^{(i)}$ may be empty (when no boundary collocation points is included in $\Omega^{(i)}$, i.e. $N_b^{(i)} = 0$). See Fig (2) for illustration. The approximate

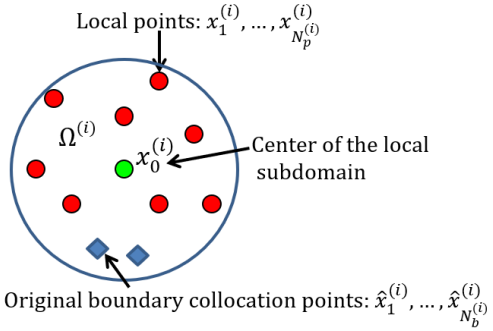

Fig. 2. Local subdomain and local points

solutions of the local problems are computed by utilizing the MFS in $\Omega^{(i)}$. To do this, define some local source points $s_1^{(i)}, s_2^{(i)}, ..., s_{N_s^{(i)}}^{(i)}$ (e.g. along the perimeter of a circle centered at $x_0^{(i)}$ with radius which is greater than $R^{(i)}$ (the definition $2R^{(i)}$ is an acceptable value). For illustration, see Fig. (3). The numbers of local sources can be kept at a common value, but $N_s^{(i)} < N_p^{(i)}$ should be satisfied. The approximate local solution is expressed in the following form:

$$u(x) := \sum_{j=1}^{N_s^{(i)}} \alpha_j^{(i)} \Phi(x - s_j^{(i)}) + \sum_{j=1}^{N_s^{(i)}} \beta_j^{(i)} \Psi(x - s_j^{(i)}), \tag{26}$$

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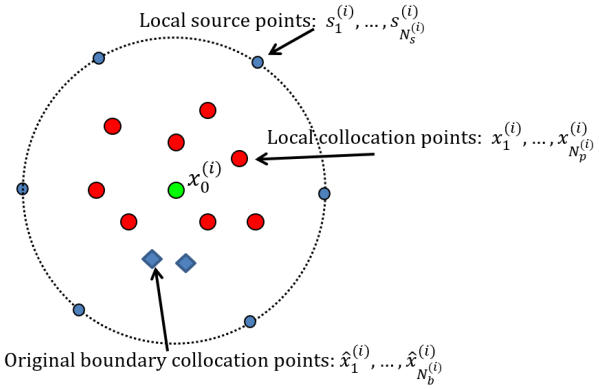

Fig. 3. Local sources along a circle

with some a priori unknown coefficients $\alpha_j^{(i)}$, $\beta_j^{(i)}$, $j = 1, 2, ..., N_s^{(i)}$. The local Navier boundary conditions are approximated by enforcing the following equalities for all local points $x_k^{(i)}$, $k = 1, 2, ..., N_p^{(i)}$:

$$\sum_{j=1}^{N_s^{(i)}} \alpha_j^{(i)} \Phi(x_k^{(i)} - s_j^{(i)}) + \sum_{j=1}^{N_s^{(i)}} \beta_j^{(i)} \Psi(x_k^{(i)} - s_j^{(i)}) = u_k^{(i)} := u(x_k^{(i)}). \tag{27}$$

If the local subdomain $\Omega^{(i)}$ contains boundary collocation points, i.e. $N_b^{(i)} > 0$, then the above equations are completed by the original boundary conditions:

$$\begin{aligned} \sum_{j=1}^{N_s^{(i)}} \alpha_j^{(i)} \Phi(\hat{x}_k^{(i)} - s_j^{(i)}) + \sum_{j=1}^{N_s^{(i)}} \beta_j^{(i)} \Psi(\hat{x}_k^{(i)} - s_j^{(i)}) &= u_0(\hat{x}_k^{(i)}), \\ \sum_{j=1}^{N_s^{(i)}} \beta_j^{(i)} \Phi(\hat{x}_k^{(i)} - s_j^{(i)}) &= w_0(\hat{x}_k^{(i)}), \end{aligned} \tag{28}$$

where $k = 1, 2, ..., N_b^{(i)}$ (since $\Delta\Phi = 0$, except for the origin, and $\Delta\Psi = \Phi$). The system (27)–(28) is overdetermined, if $2N_s^{(i)} < N_p^{(i)} + 2N_b^{(i)}$, and should be solved in the sense of least squares, i.e. by solving the corresponding Gaussian normal equations. Since the local systems are typically small, they can be expected to be only moderately ill-conditioned.

Having solved the ith local system, the approximate solution at the point $x_0^{(i)}$ is updated by:

$$u(x_0^{(i)}) := \sum_{j=1}^{N_s^{(i)}} \alpha_j^{(i)} \Phi(x_0^{(i)} - s_j^{(i)}) + \sum_{j=1}^{N_s^{(i)}} \beta_j^{(i)} \Psi(x_0^{(i)} - s_j^{(i)}), \tag{29}$$

and the iteration should be continued for all local subdomains.

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3.3 Localized solution of the biharmonic interpolation problem

Let us return to the biharmonic *interpolation* problem. Denote by S_{int} the set of predefined interpolation points: $S_{int} := \{\overline{x}_1, ..., \overline{x}_{N_{int}}\} \subset \Omega$, which is assumed to be disjoint of S. Denote by U_{int} the set of predefined values associated to the interpolation points: $U_{int} := \{\overline{u}_1, ..., \overline{u}_{N_{int}}\} \subset \Omega$ For a given central point $x_0^{(i)} \in S_0$, collect the interpolation points contained in the local subdomain $\Omega^{(i)}$:

$$S_{int}^{(i)} := S_{int} \cap \Omega^{(i)} = \{\overline{x}_1^{(i)}, ..., \overline{x}_{N_{int}^{(i)}}^{(i)}\},$$

together with the values associated to these local interpolation

$$U_{int}^{(i)} := \{\overline{u}_1^{(i)}, ..., \overline{u}_{N_{int}^{(i)}}^{(i)}\}.$$

Now the approximate local solution is expressed in the following form:

$$u(x) := \sum_{j=1}^{N_s^{(i)}} \alpha_j^{(i)} \Phi(x - s_j^{(i)}) + \sum_{j=1}^{N_s^{(i)}} \beta_j^{(i)} \Psi(x - s_j^{(i)}) + \sum_{j=1}^{N_{int}^{(i)}} \gamma_j^{(i)} \Psi(x - \overline{x}_j^{(i)}) \quad (30)$$

(the last sum plays the role of the thin plate splines).

Thus, the equalities to be enforced will be expanded as follows.

Collocation at the inner points $x_k^{(i)}$, $k = 1, 2, ..., N_p^{(i)}$:

$$\sum_{j=1}^{N_s^{(i)}} \alpha_j^{(i)} \Phi(x_k^{(i)} - s_j^{(i)}) + \sum_{j=1}^{N_s^{(i)}} \beta_j^{(i)} \Psi(x_k^{(i)} - s_j^{(i)}) + \sum_{j=1}^{N_{int}^{(i)}} \gamma_j^{(i)} \Psi(x_k^{(i)} - \overline{x}_j^{(i)}) = u(x_k^{(i)}). \quad (31)$$

Collocation at the boundary collocation points $\hat{x}_k^{(i)}$, $k = 1, 2, ..., N_b^{(i)}$:

$$\begin{aligned} \sum_{j=1}^{N_s^{(i)}} \alpha_j^{(i)} \Phi(\hat{x}_k^{(i)} - s_j^{(i)}) + \sum_{j=1}^{N_s^{(i)}} \beta_j^{(i)} \Psi(\hat{x}_k^{(i)} - s_j^{(i)}) + \\ + \sum_{j=1}^{N_{int}^{(i)}} \gamma_j^{(i)} \Psi(\hat{x}_k^{(i)} - \hat{x}_j^{(i)}) = u_0(\hat{x}_k^{(i)}) \\ \sum_{j=1}^{N_s^{(i)}} \beta_j^{(i)} \Phi(\hat{x}_k^{(i)} - s_j^{(i)}) + \sum_{j=1}^{N_{int}^{(i)}} \gamma_j^{(i)} \Phi(\hat{x}_k^{(i)} - \overline{x}_j^{(i)}) = w_0(\hat{x}_k^{(i)}). \end{aligned} \quad (32)$$

Collocation at the interpolation points $\overline{x}_k^{(i)}$, $k = 1, 2, ..., N_{int}^{(i)}$:

$$\begin{aligned} \sum_{j=1}^{N_s^{(i)}} \alpha_j^{(i)} \Phi(\overline{x}_k^{(i)} - s_j^{(i)}) + \sum_{j=1}^{N_s^{(i)}} \beta_j^{(i)} \Psi(\overline{x}_k^{(i)} - s_j^{(i)}) + \sum_{j=1}^{N_{int}^{(i)}} \gamma_j^{(i)} \Psi(\overline{x}_k^{(i)} - \overline{x}_j^{(i)}) = \\ = \overline{u}_k^{(i)}. \end{aligned} \quad (33)$$

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The system (31)–(33) is overdetermined, if $2N_s^{(i)} < N_p^{(i)} + 2N_b^{(i)}$, as earlier. It should be solved in the sense of least squares, i.e. by solving the corresponding Gaussian normal equations.

After solving the ith local system, the approximate solution at the point $x_0^{(i)}$ is updated by:

$$u(x_0^{(i)}) := \\ := \sum_{j=1}^{N_s^{(i)}} \alpha_j^{(i)} \Phi(x_0^{(i)} - s_j^{(i)}) + \sum_{j=1}^{N_s^{(i)}} \beta_j^{(i)} \Psi(x_0^{(i)} - s_j^{(i)}) + \sum_{j=1}^{N_{int}^{(i)}} \gamma_j^{(i)} \Psi(x_0^{(i)} - \overline{x}_j^{(i)}), \tag{34}$$

and the iteration should be continued for all local subdomains.

4 A numerical example

To illustrate the method, suppose that the original domain Ω is the unit circle. Let $\overline{x}_1, ..., \overline{x}_{N_{int}}$ some interpolation points defined in Ω in a quasi-random way. Let us associate the values $\overline{u}_k$ $(k = 1, 2, ..., N_{int})$ to the interpolation points by $\overline{u}_k := u^*(\overline{x}_k)$, where the test function u^* is defined by

$$u^*(x) := 1 - ||x||^2. \tag{35}$$

Note that the test function (35) itself is biharmonic, and satisfies the following boundary conditions along Γ:

$$u^*|_\Gamma = u_0^* = 0, \qquad \Delta u^*|_\Gamma = w_0^* = -4. \tag{36}$$

Therefore, it is expected that the test function can be reconstructed from its values at the interpolation points more or less exactly, provided that the number of scattered points as well as of interpolation points is large enough.

$N := N_p + N_{int} + N_b$ points were defined in $\overline{\Omega}$, where $N_p := 4000$ is the number of the inner points, while N_{int} denotes the number of interpolation points. The number of boundary collocation points N_b was set to $N_b := 500$. The inner and the interpolation points were defined also in a quasi-random way. To define the local subdomains, the radii of the subdomains were set to 0.1. In each subdomain, the number of local sources was set to the same constant denoted by N_s. Table 1 shows the discrete relative L_2-errors of the approximation

$$relative\ L_2\text{-}error := \frac{\sqrt{\sum_{j=1}^{N} \left(u(x_j) - u^*(x_j)\right)^2}}{\sqrt{\sum_{j=1}^{N} \left(u^*(x_j)\right)^2}}$$

with respect to different numbers of N_s and N_{int}.

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Table 1. The relative L_2-errors with different values of the numbers of local sources (N_s) and the numbers of interpolation points (N_{int}).

$N_s \setminus N_{int}$	250	500	1000	2000	4000
6	2.742E-3	1.964E-3	4.018E-4	1.902E-4	7.817E-5
8	1.861E-4	1.543E-4	7.558E-5	4.778E-5	1.159E-5
10	7.512E-5	1.858E-5	1.212E-5	7.193E-6	2.395E-6
12	5.649E-6	3.092E-6	1.899E-6	1.140E-6	5.217E-7

As it was expected, the relative errors decrease when the number of interpolation points increases. The errors also decrease when the number of local sources increases; note, however, that in this case, the computational cost also increases, since the local systems become larger. It is anticipated that the computational cost can be reduced by applying a simple multi-level technique.

5 Conclusions

A computational method for solving the 2D scattered data interpolation problem has been proposed. The method converts the interpolation problem to a biharmonic equation supplied with some boundary conditions along the boundary and pointwise interpolation conditions at the interpolation points. The Method of Fundamental Solutions has been generalized to this special fourth-order problem. A localization technique based on the overlapping Schwarz iteration has been also introduced. This results in an iterative algorithm; in each step, only a local subproblem is to be solved, which makes the computation simpler. In addition to it, the problem of solving large linear systems with dense and possibly ill-conditioned matrices is completely avoided.

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