Symbolic-Numeric Computation in Modeling the Dynamics of the Many-Body System TRAPPIST ⋆

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Abstract. Modeling the dynamics of the exoplanetary system TRAPPIST with seven bodies of variable mass moving around a central parent star along quasi-elliptic orbits is discussed. The bodies are assumed to be spherically symmetric and attract each other according to Newton's law of gravitation. In this case, the leading factor of dynamic evolution of the system is the variability of the masses of all bodies. The problem is analyzed in the framework of Hamiltonian's formalism and the differential equations of motion of the bodies are derived in terms of the osculating elements of aperiodic motion on quasi-conic sections. These equations can be solved numerically but their right-hand sides contain many oscillating terms and so it is very difficult to obtain their solutions over long time intervals with necessary precision. To simplify calculations and to analyze the behavior of orbital parameters over long time intervals we replace the perturbing functions by their secular parts and obtain a system of the evolutionary equations composed by 28 non-autonomous linear differential equations of the first order. Choosing some realistic laws of mass variations and physics parameters corresponding to the exoplanetary system TRAPPIST, we found numerical solutions of the evolutionary equations. All the relevant symbolic and numeric calculations are performed with the aid of the computer algebra system Wolfram Mathematica.

Keywords: Non-stationary many-body problem · Isotropic change of mass · Secular perturbations · Evolution equations · Poincaré variables.

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1 Introduction

In the wake of discoveries of exoplanetary systems [1], study of dynamic evolution of such planetary systems has become highly relevant. Observational data show that celestial bodies in such systems are non-stationary, their characteristics such as mass, size, and shape may vary with time [2]–[6]. At the same time, it is very difficult to take into account non-stationarity of the bodies because the corresponding mathematical models become very complicated. Even in the case of classical two-body problem, a general solution of which is well-known, dependence of masses on time makes the problem non-integrable; only in some special cases its exact solution can be found in symbolic form (see [7]). However, the masses of the bodies influence essentially on their interaction and motion and so it is especially interesting to investigate the dynamics of the many-body system of variable masses. One of the first works in this direction were done by T.B. Omarov [8] and J.D. Hadjidemetriou [9] (see also [10]) who started investigation of the effects of mass variability on the dynamic evolution of non-stationary gravitating systems. Later these investigations were continued in a series of works [11]–[15], where the systems of three interacting bodies with variable masses were considered. It was shown also that application of the computer algebra systems is very fruitful and enables to get new interesting results because very cumbersome symbolic computations are involved (see [16]–[18]).

It should be noted that most of the works on the dynamics of planetary systems are devoted to the study of evolution of multi-planet systems of many point bodies with constant masses. As the many-body problem is not integrable the perturbation theory based on the exact solution of the two-body problem is usually used (see [19]). This approach turned out to be very successful and many interesting results were obtained in the investigation of the motion of planet or satellite in the star-planets or double star system (see [20]–[21]). Paper [22] describes the problem of constructing a theory of four planets' motion around the central star, while the bodies masses are constant. The Hamiltonian functions are expanded into Poisson series in the osculating elements of the second Poincaré system up to the third power of the small parameter. Evolution of the planetary systems Sun - Jupiter - Saturn - Uranus - Neptune is studied in [23]. The averaged equations of motion are constructed analytically up to the third order in a small parameter for a four-planetary system. Paper [24] studies the orbital evolution of three-planet exosystem HD 39194 and the four-planet exosystems HD 141399 and HD 160691 (μ Ara). As a result, the authors have developed an averaged semi-analytical theory of motion of the second order in terms of exoplanet masses.

In the present paper, we investigate a classical problem of 8 bodies of variable masses which may be considered as model of the exoplanetary system TRAPPIST with a central star and 7 planets orbiting the star (see [1], [25]–[26]). The work is aimed at calculating secular perturbations of planetary systems on non-stationary stage of its evolution when mass variability is the leading factor of evolution. Equations of motion of the system are obtained in a general form in the relative coordinate system with the star located at the origin. The masses of

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the bodies are variable and change isotropically which means that the reactive forces do not arise. We describe main computational problems occurring when the perturbing functions are written in terms of the second Poincaré system and the evolutionary equations are obtained. For this paper, all symbolic computations were performed with the aid of the computer algebra system Wolfram Mathematica [30] which has a convenient interface and allows one to combine various kinds of computations.

The paper is organized as follows. In Section 2 we formulate the physical problem and describe the model. Then in Section 3 we derive the equations of motion in the osculating elements which are convenient for applying the perturbation theory. Section 4 is devoted to computing the perturbing functions in terms of the second Poincaré system. As a result, we obtain the evolutionary equations in Section 5 and write out them in terms of dimensionless variables. In Section 6 we describe numerical solution of the evolutionary equations. At last, we summarize the results in Conclusion.

2 Statement of the Problem and Differential Equations of Motion

Let us consider the motion of a planetary system consisting of $n + 1$ spherical bodies with isotropically changing masses mutually attracting each other according to Newton's law. Let us introduce the following notation: S is a parent star of the planetary system of mass $m_0 = m_0(t)$, P_i are planets of masses $m_i = m_i(t), (i = 1, 2, ..., n)$. We will study the motion in a relative coordinate system with the origin at the center of the parent star S the axes of which are parallel to the corresponding axes of the absolute coordinate system.

The positions of the planets are such that P_i is an inner planet relative to the P_{i+1} planets, but at the same time it is an outer planet relative to P_{i-1}. We assume that this position of the planets is preserved during the evolution.

Let the rate of mass change be different

$$\tfrac{\dot{m}_0}{m_0} \neq \tfrac{\dot{m}_i}{m_i}, \quad \tfrac{\dot{m}_i}{m_i} \neq \tfrac{\dot{m}_j}{m_j} \quad (i, j = \overline{1, n}, \quad i \neq j). \tag{1}$$

In a relative coordinate system, the equations of motion of planets with isotropically varying masses may be written as [27]–[29]

$$\ddot{\boldsymbol{r}}_i = -f \frac{(m_0 + m_i)}{r_i^3} \boldsymbol{r}_i + f \sum_{j=1}^{n}{}' m_j \left(\frac{\boldsymbol{r}_j - \boldsymbol{r}_i}{r_{ij}^3} - \frac{\boldsymbol{r}_j}{r_j^3} \right) \quad (i, j = \overline{1, n}), \tag{2}$$

where r_{ij} are mutual distances between the centers of spherical bodies

$$r_{ij} = \sqrt{(x_j - x_i)^2 + (y_j - y_i)^2 + (z_j - z_i)^2} = r_{ji}, \tag{3}$$

f is the gravitational constant, $\boldsymbol{r}_i(x_i, y_i, z_i)$ is a radius-vector of the planet P_i, and the prime sign in summation means that $i \neq j$.

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3 Equation of Motion in the Osculating Elements

3.1 Extraction of the perturbing function

Equations of motion (2) may be rewritten in the form

$$\ddot{\boldsymbol{r}}_i + f\frac{(m_0+m_i)}{r_i^3}\boldsymbol{r}_i - \frac{\ddot{\gamma}_i}{\gamma_i}\boldsymbol{r}_i = \boldsymbol{F}_i, \quad \gamma_i = \frac{m_0(t_0)+m_i(t_0)}{m_0(t)+m_i(t)} = \gamma_i(t), \tag{4}$$

where t_0 is an initial instant of time, and

$$\boldsymbol{F}_i = grad_{\boldsymbol{r}_i} W_i, \qquad W_i = W_{gi} + W_{ri}, \tag{5}$$

$$W_{gi} = f\sum_{j=1}^{n}{}' m_j\left(\frac{1}{r_{ij}} - \frac{\boldsymbol{r}_i\cdot\boldsymbol{r}_j}{r_j^3}\right), \quad \boldsymbol{r}_{ij} = \boldsymbol{r}_j - \boldsymbol{r}_i, \quad W_{ri} = -\frac{\ddot{\gamma}_i}{2\gamma_i}\boldsymbol{r}_i^2. \tag{6}$$

The equations of relative motion written in the form (4) are convenient for applying the perturbation theory developed for such non-stationary systems [6]. In the case under consideration the perturbing forces are given by the expressions (5), (6). Note that in the case of $\boldsymbol{F}_i = 0$ equations (4) reduce to integrable differential equations describing unperturbed motion of the bodies along quasi-conic sections.

3.2 Differential equations of motion in analogues of the second system of Poincaré variables

For our purposes, analogues of the second system of Poincaré canonical elements given in the works [6], [28] are preferred

$$\Lambda_i, \quad \lambda_i, \quad \xi_i, \quad \eta_i, \quad p_i, \quad q_i, \tag{7}$$

which are defined according to the formulas

$$\begin{aligned} \Lambda_i &= \sqrt{\mu_{i0}}\sqrt{a_i}, \\ \lambda_i &= l_i + \pi_i, \end{aligned} \tag{8}$$

$$\begin{aligned} \xi_i &= \sqrt{2\sqrt{\mu_{i0}}\sqrt{a_i}(1-\sqrt{1-e_i^2})}\cos\pi_i, \\ \eta_i &= -\sqrt{2\sqrt{\mu_{i0}}\sqrt{a_i}(1-\sqrt{1-e_i^2})}\sin\pi_i, \end{aligned} \tag{9}$$

$$\begin{aligned} p_i &= \sqrt{2\sqrt{\mu_{i0}}\sqrt{a_i}\sqrt{1-e_i^2}(1-\cos I_i)}\cos\Omega_i, \\ q_i &= -\sqrt{2\sqrt{\mu_{i0}}\sqrt{a_i}\sqrt{1-e_i^2}(1-\cos I_i)}\sin\Omega_i, \end{aligned} \tag{10}$$

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where

$$l_i = M_i = \tilde{n}_i[\phi_i(t) - \phi_i(\tau_i)], \qquad \pi_i = \Omega_i + \omega_i. \tag{11}$$

The differential equations of motion of n planets in the osculating analogues of the second system of Poincaré variables (8)–(11) have the canonical form

$$\begin{aligned} \dot{\Lambda}_i &= -\frac{\partial R_i^*}{\partial \lambda_i}, \quad \dot{\xi}_i = -\frac{\partial R_i^*}{\partial \eta_i}, \qquad \dot{p}_i = -\frac{\partial R_i^*}{\partial q_i}, \\ \dot{\lambda}_i &= \frac{\partial R_i^*}{\partial \Lambda_i}, \qquad \dot{\eta}_i = \frac{\partial R_i^*}{\partial \xi_i}, \qquad \dot{q}_i = \frac{\partial R_i^*}{\partial p_i}, \end{aligned} \tag{12}$$

where the Hamiltonian functions are given by

$$R_i^* = -\frac{\mu_{i0}^2}{2\Lambda_i^2} \cdot \frac{1}{\gamma_i^2(t)} - W_i\left(t, \Lambda_i, \xi_i, p_i, \lambda_i, \eta_i, q_i\right). \tag{13}$$

The canonical equations of perturbed motion (12) are convenient for describing the dynamic evolution of planetary systems when the analogues of eccentricities e_i and analogues of the inclinations I_i of the orbital plane of the planets are sufficiently small

$$e_i << 1, \qquad I_i << 1 \quad (i = \overline{1,n}). \tag{14}$$

Let us rewrite the canonical equations of motion (12) as

$$\begin{aligned} \dot{\lambda}_i &= \frac{\partial R_i^*}{\partial \Lambda_i} = \frac{\mu_{i0}^2}{\gamma_i^2 \Lambda_i^3} - \frac{\partial W_i}{\partial \Lambda_i}, \qquad & \dot{\Lambda}_i &= \frac{\partial R_i^*}{\partial \lambda_i} = \frac{\partial W_i}{\partial \lambda_i}, \\ \dot{\eta}_i &= \frac{\partial R_i^*}{\partial \xi_i} = -\frac{\partial W_i}{\partial \xi_i}, & \dot{\xi}_i &= \frac{\partial R_i^*}{\partial \eta_i} = \frac{\partial W_i}{\partial \eta_i}, \\ \dot{q}_i &= \frac{\partial R_i^*}{\partial p_i} = -\frac{\partial W_i}{\partial p_i}, & \dot{p}_i &= \frac{\partial R_i^*}{\partial q_i} = \frac{\partial W_i}{\partial q_i}. \end{aligned} \tag{15}$$

4 The Secular Part of the Main Part of the Perturbing Function

The secular part of the perturbing functions (13) has the form [29]

$$W_i^{(sec)} = W_{is}^{(sec)} + W_{ik}^{(sec)} + W_{ri}^{(sec)}. \tag{16}$$

Let us write the explicit form of the secular part of the perturbing function

$$\begin{aligned} W_i^{(sec)} &= f \sum_{s=1}^{i-1} m_s \Big(\frac{A_0^{is}}{2} + \Pi_{ii}^{is} \frac{\eta_i^2 + \xi_i^2}{2\Lambda_i} + \Pi_{is}^{is} \frac{\eta_i \eta_s + \xi_i \xi_s}{\sqrt{\Lambda_i \Lambda_s}} + \Pi_{ss}^{is} \frac{\eta_s^2 + \xi_s^2}{2\Lambda_s} - \\ & - B_1^{is} \left(\frac{p_i^2 + q_i^2}{8\Lambda_i} - \frac{p_i p_s + q_i q_s}{4\sqrt{\Lambda_i \Lambda_s}} + \frac{p_s^2 + q_s^2}{8\Lambda_s} \right) \Big) + f \sum_{k=i+1}^{n} m_k \left(\frac{A_0^{ik}}{2} + \Pi_{ii}^{ik} \frac{\eta_i^2 + \xi_i^2}{2\Lambda_i} + \right. \\ & + \Pi_{ik}^{ik} \frac{\eta_i \eta_k + \xi_i \xi_k}{\sqrt{\Lambda_i \Lambda_k}} + \Pi_{kk}^{ik} \frac{\eta_k^2 + \xi_k^2}{2\Lambda_k} - B_1^{ik} \left(\frac{p_i^2 + q_i^2}{8\Lambda_i} - \frac{p_i p_k + q_i q_k}{4\sqrt{\Lambda_i \Lambda_k}} + \left. \frac{p_k^2 + q_k^2}{8\Lambda_k} \right) \right) - \\ & - \frac{\ddot{\gamma}_i \Lambda_i^4}{2\gamma_i \mu_{i0}^2} \left(1 + \frac{3}{2\Lambda_i} \left(\xi_i^2 + \eta_i^2 \right) \right), \end{aligned} \tag{17}$$

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where the following designations are accepted for the inner planets $(s < i)$

$$\begin{aligned} \mathit{\Pi}_{ii}^{is} &= -\frac{3\alpha_{is}}{4} B_0^{is} - \frac{1}{2} B_1^{is} + \frac{15 + 6\alpha_{is}^2}{8} C_0^{is} - \frac{3\alpha_{is}}{2} C_1^{is} - \frac{9}{8} C_2^{is}, \\ \alpha_{is} &= \frac{\gamma_s a_s}{\gamma_i a_i} = \alpha_{is}(t) < 1, \end{aligned} \tag{18}$$

$$\mathit{\Pi}_{is}^{is} = \frac{1}{8}\left(9B_0^{is} + B_2^{is}\right) - \frac{9\left(1+\alpha_{is}^2\right)}{8\alpha_{is}} C_0^{is} + \frac{21}{16} C_1^{is} + \frac{3\left(1+\alpha_{is}^2\right)}{8\alpha_{is}} C_2^{is} + \frac{3}{16} C_3^{is}, \tag{19}$$

$$\begin{aligned} \mathit{\Pi}_{ss}^{is} &= -\frac{3}{4\alpha_{is}} B_0^{is} - \frac{1}{2} B_1^{is} + \frac{15\alpha_{is}^2 + 6}{8\alpha_{is}^2} C_0^{is} - \frac{3}{2\alpha_{is}} C_1^{is} - \frac{9}{8} C_2^{is}, \\ &(s < i). \end{aligned} \tag{20}$$

$$A_0^{is} = \frac{2}{\pi a_i \gamma_i} \int_0^{\pi} \frac{d\lambda}{\left(1 + \alpha_{is}^2 - 2\alpha_{is}\cos\lambda\right)^{1/2}}, \quad (s < i), \quad p = 0, 1, 2, 3, \tag{21}$$

$$\begin{aligned} B_p^{is} &= \frac{2 a_s \gamma_s}{\pi \left(a_i \gamma_i\right)^2} \int_0^{\pi} \frac{\cos(p\lambda) d\lambda}{\left(1 + \alpha_{is}^2 - 2\alpha_{is}\cos\lambda\right)^{3/2}}, \\ C_p^{is} &= \frac{2 \left(a_s \gamma_s\right)^2}{\pi \left(a_i \gamma_i\right)^3} \int_0^{\pi} \frac{\cos(p\lambda) d\lambda}{\left(1 + \alpha_{is}^2 - 2\alpha_{is}\cos\lambda\right)^{5/2}}. \end{aligned} \tag{22}$$

For the outer planets $(i < k)$ the following notations are accepted

$$\begin{aligned} \mathit{\Pi}_{ii}^{ik} &= -\frac{3\alpha_{ik}}{4} B_0^{ik} - \frac{1}{2} B_1^{ik} + \frac{15 + 6\alpha_{ik}^2}{8} C_0^{ik} - \frac{3\alpha_{ik}}{2} C_1^{ik} - \frac{9}{8} C_2^{ik}, \quad (i < k), \\ \alpha_{ik} &= \frac{\gamma_i a_i}{\gamma_k a_k} = \alpha_{ik}(t) < 1, \end{aligned} \tag{23}$$

$$\mathit{\Pi}_{ik}^{ik} = \frac{1}{8}\left(9B_0^{ik} + B_2^{ik}\right) - \frac{9\left(1+\alpha_{ik}^2\right)}{8\alpha_{ik}} C_0^{ik} + \frac{21}{16} C_1^{ik} + \frac{3\left(1+\alpha_{ik}^2\right)}{8\alpha_{ik}} C_2^{ik} + \frac{3}{16} C_3^{ik}, \tag{24}$$

$$\mathit{\Pi}_{kk}^{ik} = -\frac{3}{4\alpha_{ik}} B_0^{ik} - \frac{1}{2} B_1^{ik} + \frac{15\alpha_{ik}^2 + 6}{8\alpha_{ik}^2} C_0^{ik} - \frac{3}{2\alpha_{ik}} C_1^{ik} - \frac{9}{8} C_2^{ik}, \tag{25}$$

$$A_0^{ik} = \frac{2}{\pi a_k \gamma_k} \int_0^{\pi} \frac{d\lambda}{\left(1 + \alpha_{ik}^2 - 2\alpha_{ik}\cos\lambda\right)^{1/2}}, \quad (i < k), \quad p = 0, 1, 2, 3, \tag{26}$$

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$$B_p^{ik} = \frac{2a_i\gamma_i}{\pi\left(a_k\gamma_k\right)^2}\int\limits_0^\pi \frac{\cos(p\lambda)d\lambda}{\left(1+\alpha_{ik}^2-2\alpha_{ik}\cos\lambda\right)^{3/2}},$$
$$C_p^{ik} = \frac{2\left(a_i\gamma_i\right)^2}{\pi\left(a_k\gamma_k\right)^3}\int\limits_0^\pi \frac{\cos(p\lambda)d\lambda}{\left(1+\alpha_{ik}^2-2\alpha_{ik}\cos\lambda\right)^{5/2}}. \tag{27}$$

Note that the Laplace coefficients A_0^{ij}, B_0^{ij}, B_1^{ij}, B_2^{ij}, C_0^{ij}, C_1^{ij}, C_2^{ij}, C_3^{ij} $(i \neq j)$ are interconnected by recursive relations.

5 Evolutionary Equations

5.1 Derivation of evolution equations

The evolutionary equations that determine the behavior of the orbital parameters over long time intervals are obtained from the equations of motion if instead of the perturbing functions W_i we substitute their secular part $W_i^{(sec)}$ according to (17).

The evolution equations have the form [29]

$$\dot{\xi}_i = f\sum_{s=1}^{i-1} m_s\left(\frac{\Pi_{ii}^{is}}{\Lambda_i}\eta_i + \frac{\Pi_{is}^{is}}{\sqrt{\Lambda_i\Lambda_s}}\eta_s\right) + f\sum_{k=i+1}^{n} m_k\left(\frac{\Pi_{kk}^{ik}}{\Lambda_i}\eta_i + \frac{\Pi_{ik}^{ik}}{\sqrt{\Lambda_i\Lambda_k}}\eta_k\right) - \frac{3\ddot{\gamma}_i\Lambda_i^3}{2\gamma_i\mu_{i0}^2}\eta_i, \tag{28}$$

$$\dot{\eta}_i = -f\sum_{s=1}^{i-1} m_s\left(\frac{\Pi_{ii}^{is}}{\Lambda_i}\xi_i + \frac{\Pi_{is}^{is}}{\sqrt{\Lambda_i\Lambda_s}}\xi_s\right) - f\sum_{k=i+1}^{n} m_k\left(\frac{\Pi_{kk}^{ik}}{\Lambda_i}\xi_i + \frac{\Pi_{ik}^{ik}}{\sqrt{\Lambda_i\Lambda_k}}\xi_k\right) + \frac{3\ddot{\gamma}_i\Lambda_i^3}{2\gamma_i\mu_{i0}^2}\xi_i, \tag{29}$$

$$\dot{p}_i = -f\sum_{s=1}^{i-1} m_s B_1^{is}\left(\frac{q_i}{4\Lambda_i} - \frac{q_s}{4\sqrt{\Lambda_i\Lambda_s}}\right) - f\sum_{k=i+1}^{n} m_k B_1^{ik}\left(\frac{q_i}{4\Lambda_i} - \frac{q_k}{4\sqrt{\Lambda_i\Lambda_k}}\right), \tag{30}$$

$$\dot{q}_i = f\sum_{s=1}^{i-1} m_s B_1^{is}\left(\frac{p_i}{4\Lambda_i} - \frac{p_s}{4\sqrt{\Lambda_i\Lambda_s}}\right) + f\sum_{k=i+1}^{n} m_k B_1^{ik}\left(\frac{p_i}{4\Lambda_i} - \frac{p_k}{4\sqrt{\Lambda_i\Lambda_k}}\right), \tag{31}$$

$$\dot{\lambda}_i = \frac{\mu_{i0}^2}{\gamma_i^2\Lambda_i^3} - \frac{\partial W_i^{(sec)}}{\partial\Lambda_i}, \quad \dot{\Lambda}_i = 0. \tag{32}$$

From the second equation of the system (32) we obtain

$$\Lambda_i = const \tag{33}$$

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or

$$a_i = const. \tag{34}$$

Note that the first equation of system (32) is solved after integrating the equations (28)–(31).

5.2 Transition to dimensionless variables

Let us rewrite the evolution equations for eccentric and oblique elements in the form

$$\dot{\xi}_i = f\sum_{s=1}^{i-1} m_s\left(\frac{\Pi_{ii}^{is}}{\Lambda_i}\eta_i + \frac{\Pi_{is}^{is}}{\sqrt{\Lambda_i\Lambda_s}}\eta_s\right) + f\sum_{k=i+1}^{n} m_k\left(\frac{\Pi_{kk}^{ik}}{\Lambda_i}\eta_i + \frac{\Pi_{ik}^{ik}}{\sqrt{\Lambda_i\Lambda_k}}\eta_k\right) - \frac{3\ddot{\gamma}_i\Lambda_i^3}{2\gamma_i\mu_{i0}^2}\eta_i, \tag{35}$$

$$\dot{\eta}_i = -f\sum_{s=1}^{i-1} m_s\left(\frac{\Pi_{ii}^{is}}{\Lambda_i}\xi_i + \frac{\Pi_{is}^{is}}{\sqrt{\Lambda_i\Lambda_s}}\xi_s\right) - f\sum_{k=i+1}^{n} m_k\left(\frac{\Pi_{kk}^{ik}}{\Lambda_i}\xi_i + \frac{\Pi_{ik}^{ik}}{\sqrt{\Lambda_i\Lambda_k}}\xi_k\right) + \frac{3\ddot{\gamma}_i\Lambda_i^3}{2\gamma_i\mu_{i0}^2}\xi_i, \tag{36}$$

$$\dot{p}_i = -f\sum_{s=1}^{i-1} m_s B_1^{is}\left(\frac{q_i}{4\Lambda_i} - \frac{q_s}{4\sqrt{\Lambda_i\Lambda_s}}\right) - f\sum_{k=i+1}^{n} m_k B_1^{ik}\left(\frac{q_i}{4\Lambda_i} - \frac{q_k}{4\sqrt{\Lambda_i\Lambda_k}}\right), \tag{37}$$

$$\dot{q}_i = f\sum_{s=1}^{i-1} m_s B_1^{is}\left(\frac{p_i}{4\Lambda_i} - \frac{p_s}{4\sqrt{\Lambda_i\Lambda_s}}\right) + f\sum_{k=i+1}^{n} m_k B_1^{ik}\left(\frac{p_i}{4\Lambda_i} - \frac{p_k}{4\sqrt{\Lambda_i\Lambda_k}}\right). \tag{38}$$

Physical units: t is measured in years, a_i are measured in astronomical units, m_i are measured in masses of the Sun. In the evolution equations (35)–(38) we switch to dimensionless variables $t^*,\ a_i^*,\ m_i^*$,

$$t^* = \tau = \omega_1 t, \quad a_i^* = \frac{a_i}{a_1}, \quad m_i^* = \frac{m_i}{m_{00}}, \tag{39}$$

$$\omega_1 = \frac{\sqrt{fm_{00}}}{a_1^{3/2}} = const, \quad T_1^t = \frac{1}{\omega_1} = \frac{1}{\sqrt{fm_{00}}}a_1^{3/2} = const, \tag{40}$$

$$m_{00} = m_0(t_0) = const, \quad a_1 = a_1(t_0) = const, \quad \frac{d}{d\tau} = ()', \tag{41}$$

$$a_i = a_1 a_i^*, \quad m_i = m_{00}m_i^*. \tag{42}$$

Then we obtain

$$\Lambda_i = \sqrt{fm_{00}}\sqrt{a_1}\Lambda_i^*, \quad \Lambda_i^* = \sqrt{\mu_{i0}^*}\sqrt{a_i^*}, \quad \mu_{i0}^* = 1 + \frac{m_{i0}}{m_{00}} = const, \tag{43}$$

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$$\xi_i = \xi_i^*(fm_{00}a_1)^{1/4}, \quad \eta_i = \eta_i^*(fm_{00}a_1)^{1/4},$$
$$p_i = p_i^*(fm_{00}a_1)^{1/4}, \quad q_i = q_i^*(fm_{00}a_1)^{1/4} \tag{44}$$

$$\frac{3\ddot{\gamma}_i \Lambda_i^3}{2\gamma_i \mu_{i0}^2} = \omega_1 \frac{3\gamma_i''}{2\gamma_i}\frac{\Lambda_i^{*3}}{\mu_{i0}^{*2}}, \qquad \frac{d^2}{d\tau^2} = ()''. \tag{45}$$

Thus, the dimensionless eccentric and oblique elements have the form

$$\xi_i^* = \sqrt{2\sqrt{\mu_{i0}^*}\sqrt{a_i^*}(1-\sqrt{1-e_i^2})}\cos\pi_i,$$
$$\eta_i^* = -\sqrt{2\sqrt{\mu_{i0}^*}\sqrt{a_i^*}(1-\sqrt{1-e_i^2})}\sin\pi_i, \tag{46}$$

$$p_i^* = \sqrt{2\sqrt{\mu_{i0}^*}\sqrt{a_i^*}\sqrt{1-e_i^2}(1-\cos I_i)}\cos\Omega_i,$$
$$q_i^* = -\sqrt{2\sqrt{\mu_{i0}^*}\sqrt{a_i^*}\sqrt{1-e_i^2}(1-\cos I_i)}\sin\Omega_i. \tag{47}$$

Using the introduced notations (39)–(42) and the relations (43)–(45), we can write down the evolution equations (35)–(38) in dimensionless quantities

$$t^* = \tau, \ a_i^*, \ m_i^*, \tag{48}$$

$$\Lambda^* = const, \quad \xi_i^*, \quad \eta_i^*, \quad p_i^*, \quad q_i^*. \tag{49}$$

As a result, reducing the left and right sides of the equations (35)–(38) by a common factor

$$\omega_1(fm_{00}a_1)^{1/4} = const \tag{50}$$

we obtain the evolution equations (35)–(38) in dimensionless quantities (48)–(49). For convenience of notation, we omit the symbol $(*)$ and rewrite the equations (35)–(38) in dimensionless variables (48)–(49) in the form

$$\xi_i' = \sum_{s=1}^{i-1} m_s\left(\frac{\Pi_{ii}^{is}}{\Lambda_i}\eta_i + \frac{\Pi_{is}^{is}}{\sqrt{\Lambda_i\Lambda_s}}\eta_s\right) + \sum_{k=i+1}^{n} m_k\left(\frac{\Pi_{kk}^{ik}}{\Lambda_i}\eta_i + \frac{\Pi_{ik}^{ik}}{\sqrt{\Lambda_i\Lambda_k}}\eta_k\right) - \frac{3\gamma_i''}{2\gamma_i}\frac{\Lambda_i^3}{\mu_{i0}^2}\eta_i, \tag{51}$$

$$\eta_i' = -\sum_{s=1}^{i-1} m_s\left(\frac{\Pi_{ii}^{is}}{\Lambda_i}\xi_i + \frac{\Pi_{is}^{is}}{\sqrt{\Lambda_i\Lambda_s}}\xi_s\right) - \sum_{k=i+1}^{n} m_k\left(\frac{\Pi_{kk}^{ik}}{\Lambda_i}\xi_i + \frac{\Pi_{ik}^{ik}}{\sqrt{\Lambda_i\Lambda_k}}\xi_k\right) + \frac{3\gamma_i''}{2\gamma_i}\frac{\Lambda_i^3}{\mu_{i0}^2}\xi_i, \tag{52}$$

$$p_i' = -\sum_{s=1}^{i-1} m_s B_1^{is}\left(\frac{q_i}{4\Lambda_i} - \frac{q_s}{4\sqrt{\Lambda_i\Lambda_s}}\right) - \sum_{k=i+1}^{n} m_k B_1^{ik}\left(\frac{q_i}{4\Lambda_i} - \frac{q_k}{4\sqrt{\Lambda_i\Lambda_k}}\right), \tag{53}$$

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$$q_i' = \sum_{s=1}^{i-1} m_s B_1^{is} \left(\frac{p_i}{4\Lambda_i} - \frac{p_s}{4\sqrt{\Lambda_i \Lambda_s}} \right) + \sum_{k=i+1}^{n} m_k B_1^{ik} \left(\frac{p_i}{4\Lambda_i} - \frac{p_k}{4\sqrt{\Lambda_i \Lambda_k}} \right). \quad (54)$$

At the same time, the expressions

$$\Pi_{ii}^{is}, \quad \Pi_{is}^{is}, \quad \Pi_{kk}^{ik}, \quad \Pi_{ik}^{ik} \quad (55)$$

in the equations (51)–(54) and the Laplace coefficients keep their form, according to the formulas (18)–(22) and (23)–(27). They become dimensionless quantities.

6 The Algorithm of Calculations

In our model, it is more convenient to use analogues of the second system of Poincaré canonical elements [9] and to write the equations of motion in the form (15). The secular part of the perturbing functions W_i is defined in the form (16)–(20), (23)–(25) with the Laplace coefficients of the form (21), (22), (26), (27). The evolutionary equations are written in dimensionless variable in the form (51)–(54).

I. We define and study a system of 14 differential equations of the form (53)–(54). To do this, we perform the following steps:

a) Determine the type of change in the masses of the central star and planets (we consider dependencies accordingly Eddington-Jeans law)

$$m_0(t) = (\varepsilon_0(1-n_0)(t-t_0) + m_{00}^{1-n_0})^{1/(1-n_0)} \quad (n_0 = 3),$$

$$m_i(t) = (\varepsilon_i(1-n_i)(t-t_0) + m_{i0}^{1-n_i})^{1/(1-n_i)} \quad (n_i = 2, \quad i = \overline{1,7});$$

b) taking into account the type of functions α_i ($i = \overline{1,7}$) we compute the Laplace coefficients B_1^{ik} from (27) ($p = 1$);

c) for each of the seven planets we build a system of two differential equations of the form (53)–(54) and add the initial conditions (47),

$$\begin{gathered} \sigma_1 = 1.374, \ \nu_1 = 10^{-5}, \ a_1 = 0.01154, \ e_1 = 0.00622, \\ \pi_1(t_0) = 21^\circ, \ I_1(t_0) = 0.35^\circ, \ \Omega_1(t_0) = 45^\circ \end{gathered} \quad (56)$$

for the first planet P_1. Similar initial conditions at the point $\tau = 0$ for the other six planets $P_2, ..., P_7$ can be taken from [1]. Adding these initial conditions to the system of differential equations, we obtain the required system (53)–(54);

d) using the numerical integration, we find the functions p_i, q_i, and then visualize the orbital elements (see Fig. 1)

$$\sin^2 I_j \approx \tfrac{p_j^2 + q_j^2}{\Lambda_j} \quad (j = \overline{1,7}). \quad (57)$$

II. We define and study a system of 14 differential equations of the form (51)–(52). To do this, we perform the following steps:

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a) Using the results of step I a) we determine the Laplace coefficients A_0^{ij}, B_0^{ij}, B_1^{ij}, B_2^{ij}, C_0^{ij}, C_1^{ij}, C_2^{ij}, C_3^{ij} from the system (21), (22), (26), (27);

b) on the next step we define functions Π_{ii}^{ij} from (18)–(20) and (23)–(25);

c) for each of the seven planets we build a system of two differential equations (51)–(52), add the initial conditions (46) and other initial conditions from [1] at the point $\tau = 0$. Adding initial conditions to the system of differential equations, we obtain the required system (51)–(52);

d) using the numerical integration, we find the functions ξ_i, η_i, and then eccentric elements $e_i^2 \approx \frac{\xi_i^2+\eta_i^2}{\Lambda_i}$ (Fig. 2), $\pi_i = -\arctan\frac{\eta_i}{\xi_i}$;

e) performing steps I d), II d) we can find and visualize the orbital elements $\omega_i = \pi_i - \Omega_i$.

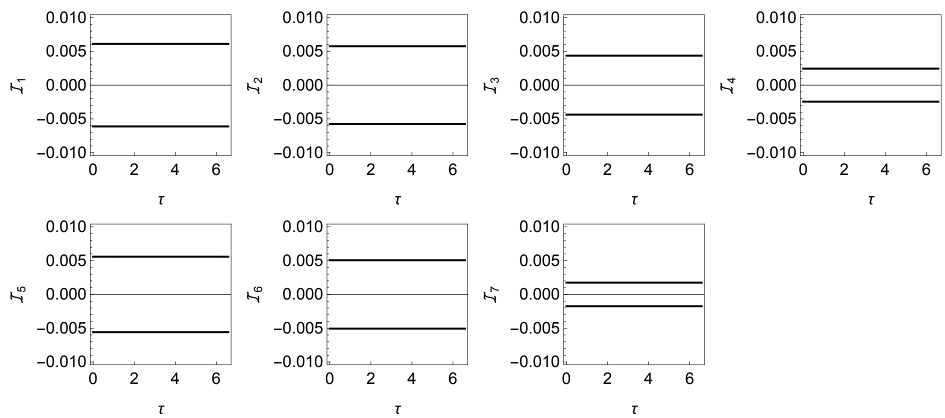

Fig. 1. Plots of the functions $\sin^2(I_i) = \frac{p_i^2+q_i^2}{\Lambda_i}$ $(i = \overline{1,7})$ (see (57))

7 Conclusion

For the first time, a system of differential equations has been obtained that describes the motion of planets of variable masses around a central star in the TRAPPIST-1 system.

The evolution equations have been obtained in dimensionless variables, and their numerical solutions have been found in the case of the physical parameters corresponding to the TRAPPIST-1 system. The calculation time was chosen to be 2000 revolutions of the first planet.

Numerical experiment was carried out in three ways: first, the Laplace coefficients in elliptic functions were directly calculated; then (in the second and third cases) these coefficients were expanded into series up to the 4th and 2nd order of smallness. The results of the calculations practically coincided. The computation time has significantly decreased in the second case in comparison to the first one, and in the third case it has decreased even more significantly.

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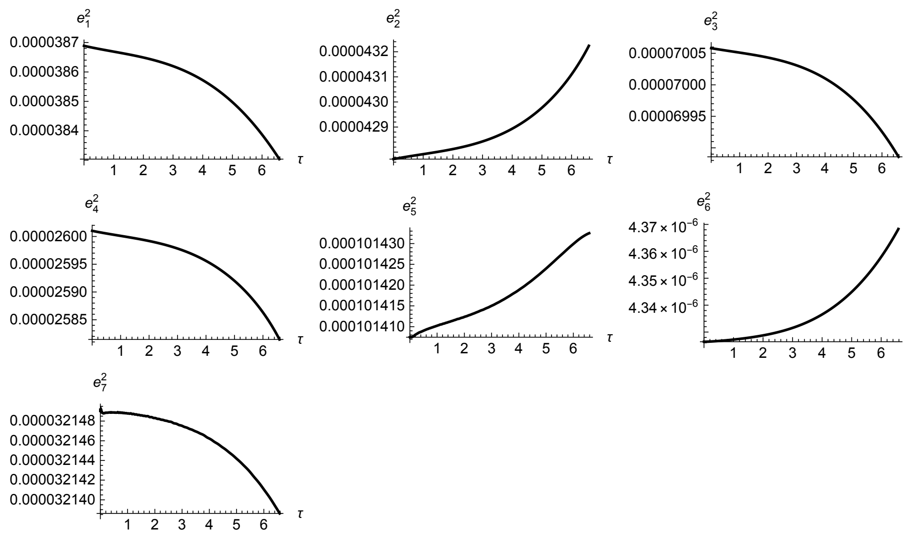

Fig. 2. Plots of the functions e_i^2 $(i = \overline{1,7})$

The graph (Fig. 1) shows the time dependence of the inclination of the planets orbits determined by the formulas (57). They change in a very narrow range which allows us to conclude that the system is moving in one plane (the Laplace plane). At the same time the changes of eccentricities of planets' orbits are more noticeable (see Fig. 2).

Note that choosing different laws of mass variations, one can find numerical solutions to the evolutionary equations for different values of physical parameters. Such simulation enables to investigate dynamical evolution of the exoplanetary system and to understand better an influence of the masses change on the system motion. The corresponding calculations may be carried out with the aid on any software but here we used the computer algebra system Wolfram Mathematica which enables to visualize easily the obtained results, as well.

References

1. NASA Exoplanet Exploration, https://exoplanets.nasa.gov/. Last accessed 28 April 2022
2. Omarov, T.B. (Ed.): Non-Stationary Dynamical Problems in Astronomy. Nova Science Publ. Inc., New-York (2002)
3. Bekov, A.A., Omarov, T.B.: The theory of orbits in non-stationary stellar systems. Astronomical and Astrophysical Transactions **22**(2), 145–153 (2003). https://doi.org/10.1080/1055679031000084803
4. Eggleton, P.: Evolutionary processes in binary and multiple stars. Cambridge University Press, New York (2006)
5. Luk'yanov, L.G.: Dynamical evolution of stellar orbits in close binary systems with conservative mass transfer. Astronomy Reports **52**, 680–692 (2008)

ICCS Camera Ready Version 2023
To cite this paper please use the final published version:
DOI: 10.1007/978-3-031-36024-4_36

6. Minglibayev, M.Zh.: Dynamics of gravitating bodies with variable masses and sizes [Dinamika gravitiruyushchikh tel s peremennymi massami i razmerami]. LAMBERT Academic Publ., Saarbrucken (2012)
7. Berkovič, L.M.: Gylden-Meščerski problem. Celestial Mechanics **24**, 407–429 (1981)
8. Omarov, T.B.: Two-body problem with corpuscular radiation. Sov. Astron. **7**, 707–714 (1963)
9. Hadjidemetriou, J.D.: Two-body problem with variable mass: A new approach. Icarus **2**, 440–451 (1963). https://doi.org/10.1016/0019-1035(63)90072-1
10. Veras, D., Hadjidemetriou, J.D., Tout, C.A.: An Exoplanet's Response to Anisotropic Stellar Mass-Loss During Birth and Death. Monthly Notices of the Royal Astronomical Society **435**(3), 2416–2430 (2013). https://doi.org//10.1093/mnras/stt1451
11. Minglibayev, M.Zh., Mayemerova, G.M.: Evolution of the orbital-plane orientations in the two-protoplanet three-body problem with variable masses. Astronomy Reports **58**(9), 667–677 (2014). https://doi.org/10.1134/S1063772914090066
12. Prokopenya, A.N., Minglibayev, M.Zh., Beketauov, B.A.: Secular perturbations of quasi-elliptic orbits in the restricted three-body problem with variable masses. International Journal of Non-Linear Mechanics **73**, 58–63 (2015). https://doi.org/10.1016/j.ijnonlinmec.2014.11.007
13. Minglibayev, M.Zh., Prokopenya, A.N., Mayemerova, G.M., Imanova, Zh.U.: Three-body problem with variable masses that change anisotropically at different rates. Mathematics in Computer Science **11**, 383–391 (2017). https://doi.org/10.1007/s11786-017-0306-4
14. Prokopenya, A.N., Minglibayev, M.Zh., Mayemerova, G.M., Imanova, Zh.U.: Investigation of the restricted problem of three bodies of variable masses using computer algebra. Programming and Computer Software **43**(5) 289–293 (2017). https://doi.org/10.1134/S0361768817050061
15. Minglibayev, M., Prokopenya, A., Shomshekova, S.: Computing perturbations in the two-planetary three-body problem with masses varying non-isotropically at different rates. Mathematics in Computer Science **14**(2), 241–251 (2020). https://doi.org/10.1007/s11786-019-00437-0
16. Prokopenya, A.N., Minglibayev, M.Zh., Mayemerova, G.M.: Symbolic computations in studying the problem of three bodies with variable masses. Programming and Computer Software **40**(2), 79–85 (2014). https://doi.org/10.1134/S036176881402008X
17. Minglibayev, M., Prokopenya, A., Shomshekova, S.: Applications of computer algebra in the study of the two-planet problem of three bodies with variable masses. Programming and Computer Software **45**(2), 73–80 (2019). https://doi.org/10.1134/S0361768819020087
18. Prokopenya, A.N., Minglibayev, M.Zh., Baisbayeva, O.: Analytical computations in studying translational-rotational motion of a non-stationary triaxial body in a central gravitational field. In: F. Boulier, M. England, T.M. Sadykov, E.V. Vorozhtsov (Eds.) Computer Algebra in Scientific Computing / CASC'2020. LNCS, vol. 12291, pp. 478–491. Springer, Heidelberg (2020). https://doi.org/10.1007/978-3-030-60026-6_28
19. Murray, C.D., Dermott, S.F.: Solar System Dynamics. Cambridge University Press, New York (1999)
20. Lidov, M.L., Vashkov'yak, M.A.: On quasi-satellite orbits in a restricted elliptic three-body problem. Astronomy Letters **20**(5), 676–690 (1994)
21. Ford, E.B., Kozinsky, B., Rasio, F.A.: Secular evolution of hierarchical triple star systems. The Astronomical Journal **535**, 385–401 (2000)

ICCS Camera Ready Version 2023
To cite this paper please use the final published version:
DOI: 10.1007/978-3-031-36024-4_36

22. Perminov, A.S., Kuznetsov, E.D.: The implementation of Hori–Deprit method to the construction averaged planetary motion theory by means of computer algebra system Piranha. Mathematics in Computer Science **14**(2), 305–316 (2020). https://doi.org/10.1007/s11786-019-00441-4
23. Perminov, A.S., Kuznetsov, E.D.: The orbital evolution of the Sun–Jupiter–Saturn–Uranus–Neptune system on long time scales. Astrophysics and Space Science **365**(8), 144 (2020). https://doi.org/10.1007/s10509-020-03855-w
24. Perminov, A.S., Kuznetsov E.D.: Orbital evolution of the extrasolar planetary systems HD 39194, HD 141399, and HD 160691. Astronomy Reports **96**(10), 795–813 (2019). https://doi.org/10.1134/S1063772919090075
25. Gillon, M. et.al.: Seven temperate terrestrial planets around the nearby ultracool dwarf star TRAPPIST-1. Nature **542**(7642), 456–460 (2017). https://doi.org/10.1038/nature21360
26. Shallue, C. J., Vanderburg, A.: Identifying exoplanets with deep learning: A five-planet resonant chain around Kepler-80 and an Eighth planet around Kepler-90. The Astronomical Journal **155**(2), 94 (2018). https://doi.org/10.3847/1538-3881/aa9e09
27. Minglibayev, M.Zh., Kosherbayeva, A.B.: Differential equations of planetary systems. Reports of the National Academy of Sciences of the Republic of Kazakhstan **2**(330), 14–20 (2020). https://doi.org/10.32014/2020.2518-1483.26
28. Minglibayev, M.Zh., Kosherbayeva, A.B.: Equations of planetary systems motion. News of the National Academy of Sciences of the Republic of Kazakhstan. Physico-Mathematical Series **6**(334), 53–60 (2020). https://doi.org/10.32014/2020.2518-1726.97
29. Prokopenya, A.N., Minglibayev, M.Zh., Kosherbayeva A.B.: Derivation of evolutionary equations in the many–body problem with isotropically varying masses using computer algebra. Programming and Computer Software **48**(2), 107–115 (2022). https://doi.org/10.1134/S0361768822020098
30. Wolfram, S.: An elementary introduction to the Wolfram Language. Second Ed. Wolfram Media, New York (2016)

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