PIES in multi-region elastic problems including body forces

Agnieszka Bołtuć[0000−0002−4761−1170] and Eugeniusz Zieniuk[0000−0002−6395−5096]

University of Bialystok, Institute of Computer Science, Bialystok, Poland
{a.boltuc,e.zieniuk}@uwb.edu.pl

Abstract. The paper presents the formulation of a parametric integral equation system (PIES) for boundary problems with piecewise homogeneous media and body forces. The multi-region approach is used, in which each region is treated separately and modeled globally by a Bezier surface. Each subarea can have different material properties, and different body loads can act on it. Finally, they are connected by dedicated conditions. Two examples are solved to confirm the effectiveness of the proposed approach. The results are compared with analytical solutions and those received from other numerical methods (FEM, BEM).

Keywords: PIES · Multi-region analysis · Body forces · Bezier surfaces.

1 Introduction

In many elasticity problems, the considered body is made up of different materials or mechanical material properties (e.g. Young's modulus, Poisson's ratio) vary in some piecewise fashion. The approach to this type of problem differs significantly in the best-known numerical methods for solving them. The oldest and most popular finite element method (FEM) [1, 2] is characterized by the general strategy regardless of the problem. Therefore, the whole body is always divided into finite elements, for which the same or various material properties can be posed. It can be said that different materials are taken into account automatically. On the other hand, the number of required elements and nodes is the largest here. The boundary element method (BEM) [3, 4] reduces the problem size, because modeling is limited to the boundary only. Bodies in which material properties vary piecewise are approximated here by a system of homogeneous bodies. Such an approach is called multi-region formulation. However, the methods based on the boundary integral generate the dense resulting matrix, while in FEM, it is sparse. The method developed by the authors, called the parametric integral equation system (PIES), also applies to bodies made up of subdomains with different material properties [5]. PIES significantly reduces the number of input data necessary for modeling the shape, because only the boundary is created using parametric curves. Thus, discretization into elements is eliminated.

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But what about problems with piecewise constant material properties in which body forces also appear? The approach in FEM does not change, because various forces can be posed on different finite elements. In BEM, the domain should be created. This process is technically very similar to discretization in FEM, but the used elements are called cells. There are some body forces for which only the boundary can be defined, because the domain integral is transformed into the boundary. However, the general approach requires the application of cells for each region separately. PIES has been used to solve problems with body forces, but only in homogeneous domains [6, 7]. It significantly simplifies the way of modeling, because the whole area on which the body forces act is created using a single Bezier surface of any degree. This, in turn, is reduced to just setting control points.

This paper presents PIES for multi-region elastic problems, but also including body forces. Each considered region with different material properties and various body forces is modeled globally using the Bezier surface and its control points. Then they are connected by the compatibility and equilibrium conditions at the common interface. PIES formula for such problems is developed together with a numerical solution scheme. Two examples are solved, confirming the approach's effectiveness and accuracy.

2 PIES for elasticity with body forces

The isotropic linear elastic solids with body forces are considered. The governing equations, known as Navier's equations, are expressed by

$$\mu u_{i,jj} + \frac{\mu}{1-2\nu} u_{j,ji} + b_i = 0, \tag{1}$$

where μ is the shear modulus, ν is the Poisson's ratio, u_i is the displacement,b_i is the body force and commas represent differentiation with respect to spatial coordinates ($i, j = 1, 2$ for 2D).

The equation (1) can be transformed into the corresponding integral equation using the strategy described in [8]

$$\begin{aligned} 0.5\boldsymbol{u}_l(\bar{s}) = \sum_{j=1}^{n} \int_{s_{j-1}}^{s_j} \left\{ \mathbf{U}^*_{lj}(\bar{s}, s)\boldsymbol{p_j}(s) - \mathbf{P}^*_{lj}(\bar{s}, s)\boldsymbol{u_j}(s) \right\} J_j(s) ds \\ + \sum_{k=1}^{m} \int_{\Omega_k} \bar{\boldsymbol{U}}^*_l(\bar{s}, \boldsymbol{y})\boldsymbol{b_k}(\boldsymbol{y}) d\Omega(\boldsymbol{y}), \end{aligned} \tag{2}$$

where $J_j(s)$ is the Jacobian of transformation to the parametric reference system, $l, j = 1..n$, $s_{l-1} \leq \bar{s} \leq s_l$, $s_{j-1} \leq s \leq s_j$ and n, m are the number of segments and subregions.

Functions $\boldsymbol{u_j}(s)$, $\boldsymbol{p_j}(s)$ describe the distribution of displacements and tractions on the boundary, respectively. On each segment, one is prescribed, and the other is to be solved. The function $\boldsymbol{b}(\boldsymbol{y})$ represents the vector of body forces.

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The fundamental solutions for displacement $\mathbf{U}^*_{lj}(\bar{s},s)$ and traction $\mathbf{P}^*_{lj}(\bar{s},s)$ are presented explicitly in [8]. The solution $\bar{\boldsymbol{U}}^*_l(\bar{s},\boldsymbol{y})$ can be found in [6].

PIES' boundary and domain are defined in a parametric reference system using well-known computer graphics tools like curves and surfaces [9]. They are analytically incorporated into the formalism of PIES by functions η [8], which represent the distance between two boundary/domain points.

The collocation method [10] is used for the PIES solution. Unknown boundary functions are approximated by series with arbitrary basis functions, e.g., Legendre or Chebyshev polynomials [6–8]. The number of expressions in the series affects the accuracy of the solutions. Only this parameter should be increased to reduce the error, without interfering with the shape and re-discretization. After substituting approximation series to (2) and writing the resulting equation for all collocation points, the PIES matrix form is obtained

$$\boldsymbol{H}\boldsymbol{u} = \boldsymbol{G}\boldsymbol{p} + \boldsymbol{b}, \tag{3}$$

where $\boldsymbol{H}$, $\boldsymbol{G}$ are square matrices of boundary integrals from (2), while $\boldsymbol{b}$ is the vector of integrals over the domain.

After solving equation (3) only the boundary solutions are obtained. To calculate results within the domain, the integral identities for displacements and stresses are used. They are presented in [5, 8] (without body forces).

3 Multi-region formulation

As mentioned in previous sections, for bodies with piecewise homogeneous materials, it is necessary to consider more than one region. Then such a body can be approximated by a system of homogeneous bodies with different material constants. To illustrate the problem, consider a region Ω consisting of two subregions Ω_1 and Ω_2. They are separated by an interface boundary Γ_I and surrounded by respectively Γ_1 and Γ_2. Each region has different mechanical material properties and different body forces can act on it (Fig. 1). The analysis of such a problem consists in considering each region separately [3], which results in the following matrix equations for Ω_1 and Ω_2

$$\begin{bmatrix}\boldsymbol{H}_1 & \boldsymbol{H}_1^I\end{bmatrix}\begin{bmatrix}\boldsymbol{u}_1\\ \boldsymbol{u}_1^I\end{bmatrix} = \begin{bmatrix}\boldsymbol{G}_1 & \boldsymbol{G}_1^I\end{bmatrix}\begin{bmatrix}\boldsymbol{p}_1\\ \boldsymbol{p}_1^I\end{bmatrix} + \{\boldsymbol{b}_1\}, \tag{4}$$

$$\begin{bmatrix}\boldsymbol{H}_2 & \boldsymbol{H}_2^I\end{bmatrix}\begin{bmatrix}\boldsymbol{u}_2\\ \boldsymbol{u}_2^I\end{bmatrix} = \begin{bmatrix}\boldsymbol{G}_2 & \boldsymbol{G}_2^I\end{bmatrix}\begin{bmatrix}\boldsymbol{p}_2\\ \boldsymbol{p}_2^I\end{bmatrix} + \{\boldsymbol{b}_2\}, \tag{5}$$

where $\boldsymbol{H}_1$, $\boldsymbol{G}_1$ contains the boundary integrals over Γ_1, $\boldsymbol{H}_2$, $\boldsymbol{G}_2$ over Γ_2, while $\boldsymbol{H}_1^I$, $\boldsymbol{G}_1^I$ over the interface boundary Γ_I from Ω_1 and $\boldsymbol{H}_2^I$, $\boldsymbol{G}_2^I$ over Γ_I from Ω_2.

The equations (4) and (5) are connected by compatibility and equilibrium conditions at the interface boundary Γ_I

$$\boldsymbol{u}_1^I = \boldsymbol{u}_2^I = \boldsymbol{u}^I, \boldsymbol{p}_1^I = \boldsymbol{p}_2^I = \boldsymbol{p}^I. \tag{6}$$

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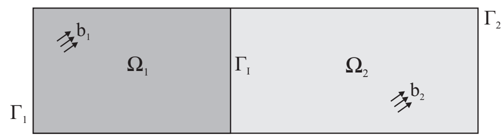

Fig. 1. Multiregion body.

The inclusion of (6) in (4) and (5) results in the following matrix equation

$$\begin{bmatrix} \boldsymbol{H}_1 & \boldsymbol{H}_1^I & -\boldsymbol{G}_1^I & 0 \\ 0 & \boldsymbol{H}_2^I & \boldsymbol{G}_2^I & \boldsymbol{H}_2 \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}^I \\ \boldsymbol{p}^I \\ \boldsymbol{u}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{G}_1 & 0 \\ 0 & \boldsymbol{G}_2 \end{bmatrix} \begin{bmatrix} \boldsymbol{p}_1 \\ \boldsymbol{p}_2 \end{bmatrix} + \begin{Bmatrix} \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \end{Bmatrix}. \tag{7}$$

Matrices in (7) are block-banded, and each block corresponds to one region. The matrix on the left contains overlaps between blocks at the common interface.

When calculating solutions in the domain, the separate identity is required for each region. Which one is used depends on the region of the examined point.

4 Modeling of regions

The way of modeling problems with various material properties strongly depends on the method used. In FEM [1, 2], the procedure is general and does not differ from the case where the material properties are constant over the whole area. It comes from the fact that the domain is divided into finite elements on which various properties can be applied (Fig. 2a). BEM [3, 4] proposes two approaches: modeling only the boundary of separate regions (if the integral over the domain is transformed to the boundary) by boundary elements (Fig. 2b) or modeling the boundary and the domains of the regions using cells (Fig. 2c). Both approaches could be implemented in PIES, but the way of modeling the shape is entirely different, because no division into elements or cells is required.

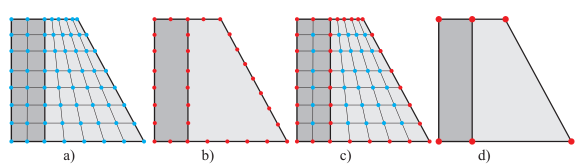

Fig. 2. Modeling in a) FEM, b) BEM (only the boundary), c) BEM (the boundary and the domain), d) PIES.

As shown in Fig. 2d, in PIES, each region is modeled by a separate Bezier surface [9]. They can be of various degrees, e.g., the bilinear surface, used for polygonal shapes, requires only 4 corner points to be given. For curved region

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shapes, surfaces of higher degrees are used. However, the third-degree patch has sufficient design flexibility, and higher degrees require longer processing time. It consists of 12 control points for modeling the boundary and 4 responsible for the shape in 3D (in 2D problems, they are not essential). However, the formalism of PIES allows the use of surfaces other than Bezier.

Comparing the approaches presented in Fig. 2 on the sample shape and discretization schemas, it can be seen that PIES significantly reduces the set of required nodes (from many elements/cells in FEM/BEM to a few corner points). In PIES, the accuracy depends on the number of expressions in the approximation series, not the number of data used for modeling. For the same shape, always minimal data set is required. It comes from the fact that the shape approximation is separated from the approximation of the solutions.

5 Tested examples

5.1 Example 1

The first problem concerns elastic analysis with a centrifugal load. A square plate (Fig. 3) rotates about the x-axis with angular velocity $\omega = 100rad/s$. There is a discontinuity in the density distribution and material properties: $\rho_1 = 1 \times 10^{-6}$, $E = 210GPa$, $v = 0.2$ for $0 \leq y \leq 50$ and $\rho_2 = 2 \times 10^{-6}$, $E = 160GPa$, $v = 0.3$ for $50 < y \leq 100$.

As seen in Fig. 3, the problem in PIES is modeled using two bilinear Bezier surfaces, one for each region. They have been defined by 6 control (corner) points. Boundary functions (u and p) are approximated by series with Chebyshev polynomials of the first kind with 7 expressions for each boundary segment.

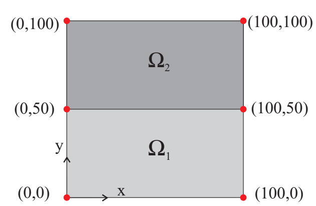

Fig. 3. Square plate with two materials rotating about the x-axis.

The exact solution for the one-dimensional problem is given in [11] by

$$\begin{gathered}\sigma_y = \frac{\rho_1\omega^2}{8}\left[2L(L-2y)-(L-2y)^2\right]+\rho_2\omega^2\frac{3L^2}{8}, 0 \leq y \leq 50,\\ \sigma_y = \rho_2\omega^2\left[L(L-y)-\frac{(L-y)^2}{2}\right], 50 < y \leq 100.\end{gathered} \quad (8)$$

The considered geometry in other numerical methods requires posing elements, e.g., in [11] 80 constant boundary elements. In this paper, the FEM model is used for comparison purposes. Two meshes are applied, with 400 and

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1600 8-noded quadrilateral finite elements. Finally, the number of solved equations is 112 in PIES and 2562/9922 in FEM (depending on the mesh).

Figure 4 shows the comparison of stress distribution at $x = 50$. Only FEM results with a finer mesh are presented, as the average relative error obtained at the considered cross-section equals 1.54% (for a coarser mesh it is 1.6%). For the PIES method, the error equals 1.02%. Additionally, the computational times are compared. PIES solved the problem in $0.772s$, while FEM in $2.05s$.

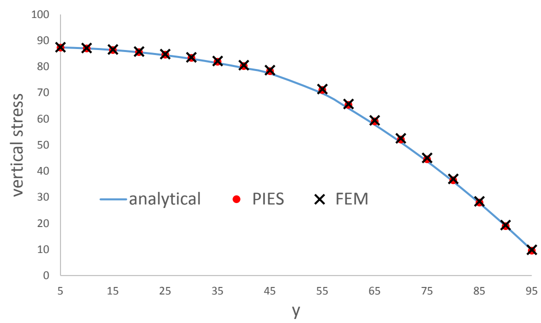

Fig. 4. Stress σ_y distribution at $x = 50$.

5.2 Example 2

The second example concerns a footing on a horizontal layer of soil (Fig. 5a) under a uniform compressive load with a magnitude of 1 ton/m and a self-weight (of the footing and soil). The considered material properties are: $E = 1 \times 10^4$ ton/m^2, $v = 0.4$, $\gamma = 2$ ton/m^3 for the soil, and $E = 2 \times 10^6$ ton/m^2, $v = 0.2$, $\gamma = 2.4$ ton/m^3 for the footing.

As shown in Fig. 5b, modeling both regions in PIES using 2 bilinear Bezier surfaces requires posing only 8 corner points. Approximation series for u and p for each boundary segment contains 6 expressions with Chebyshev polynomials of the first kind used as basis functions.

The same problem can be defined in other well-known methods like FEM or BEM. However, the number of elements and nodes is much higher than the number of corner points applied in PIES. For example, in the model created in BEM, 75 quadratic boundary elements are used, while FEM requires 243 8-noded finite elements.

The first test concerns the analysis of displacements u_y at the top boundary of the region Ω_2. Table 1 presents the results obtained by PIES after solving the system of 144 equations, by BEM with 362 equations and FEM with 1612 equations. The vertical stress values for both regions at $x = 0$ are also obtained and compared with BEM and FEM (Fig. 6).

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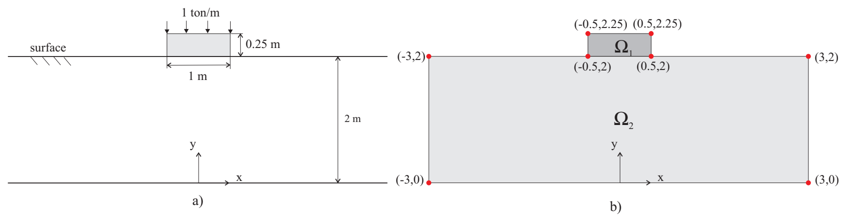

Fig. 5. Footing on a horizontal layer: a) definition, b) modeling.

Table 1. Vertical displacements along the boundary $y = 2$.

x	FEM	BEM	PIES
3	-0.003278	-0.003277	-0.003277
2.75	-0.003279	-0.003279	-0.003278
2.5	-0.003283	-0.003281	-0.003283
2.25	-0.003292	-0.003290	-0.003291
2	-0.003305	-0.003306	-0.003305
1.75	-0.003326	-0.003327	-0.003326
1.5	-0.003359	-0.003357	-0.003360
1.25	-0.003409	-0.003402	-0.003411
1	-0.003489	-0.003480	-0.003490
0.75	-0.003623	-0.003622	-0.003621
0.5	-0.003964	-0.003876	-0.003915

The PIES solutions presented in Table 1 and Fig. 6 are very close to FEM and BEM results. In element methods, however, a much larger number of input data necessary for modeling and the number of equations in the solved system are observed. The evaluated computational times are $1.43s$ for FEM and $0.954s$ for PIES. For BEM, it is less than $1s$, but accurate reading is impossible.

6 Conclusions

The approach for solving multi-region problems with piecewise constant material properties and body forces is derived in this paper. PIES equation is created for each region, and then they are connected using compatibility and equilibrium conditions at the interface boundaries. Domain modeling is required due to the presence of body forces, but each region is globally defined with a surface patch. Therefore, dividing them into elements or cells is eliminated.

The proposed formulation is tested on two examples: centrifugal and gravitational loads. The results are compared with exact and numerical solutions (FEM, BEM). They agree (or even are more accurate) with a significantly smaller number of data for modeling and solving the problem. Moreover, the computational time in the examined cases also proves in favor of PIES.

The limitation of the method can be a very complex shape that cannot be modeled with a single Bezier patch. Then different types of surfaces that are more flexible and can define domains other than quadrangular should be applied. This is one of the future research goals. It is also planned to use the method to solve

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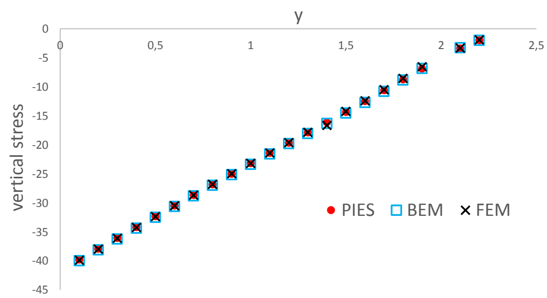

Fig. 6. Vertical stresses in the cross-section $x = 0$.

elastoplastic problems with piecewise constant material properties and then also with body forces. In addition, in this paper, authors did not analyze singularities that occur in problems with multiple domains due to, e.g., reentrant corners. This issue should also be the subject of a detailed examination.

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