Solving uncertainly defined curvilinear potential 2D BVPs by the IFPIES

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Abstract. The paper presents the interval fast parametric integral equations system (IFPIES) applied to model and solve uncertainly defined curvilinear potential 2D boundary value problems with complex shapes. Contrary to previous research, the IFPIES is used to model the uncertainty of both boundary shape and boundary conditions. The IFPIES uses interval numbers and directed interval arithmetic with some modifications previously developed by the authors. Curvilinear segments in the form of Bézier curves of the third degree are used to model the boundary shape. However, the curves also required some modifications connected with applied directed interval arithmetic. It should be noted that simultaneous modelling of boundary shape and boundary conditions allows for a comprehensive approach to considered problems. The reliability and efficiency of the IFPIES solutions are verified on 2D complex potential problems with curvilinear domains. The solutions were compared with the interval solutions obtained by the interval PIES. All performed tests indicated the high efficiency of the IFPIES method.

Keywords: Interval fast parametric integral equations system · Interval numbers · Directed interval arithmetic · Uncertainty.

1 Introduction

The interval fast parametric integral equations system (IFPIES) [1] is a robust numerical tool for solving uncertainly defined boundary value problems (BVPs). It is based on successors of the original parametric integral equations system (PIES) such as the interval parametric integral equations system (IPIES) [2] and the fast parametric integral equations system (FPIES) [3].

The IPIES was developed to solve uncertainly defined problems. In traditional modelling and solving BVPs, the shape of the boundary, boundary conditions and some other parameters of the considered domain (i.e. material properties) are defined precisely using real numbers. In practice, we should measure some physical quantities to obtain these data. However, the accuracy of determining the physical quantity is affected by, e.g. gauge reading error, inaccuracy of measurement instruments or approximations of the models used in the analysis

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of measurements. Therefore, we should consider the uncertainty of the domain description in modelling and solving BVPs.

Classical mathematical models require exact values of the data. Therefore, the direct consideration of uncertainty is not possible. However, many known methods were modified to consider uncertainty (e.g. [4–6]). Some of them applied interval numbers and interval arithmetic to the methods of modelling and solving BVPs. Therefore, the interval finite element method (IFEM) [7], the interval boundary element method (IBEM) [8], and the IPIES were obtained. However, either the IFEM or the IBEM considered only the uncertainty of material parameters or boundary conditions. Only in a few papers some parameters of the boundary shape (e.g. radius or beam length) were uncertainly defined. Therefore, the possibility of simultaneous consideration of all uncertainties mentioned above in the IPIES [2] becomes a significant advantage.

Although the IPIES has other advantages (e.g., defining the boundary by curves widely used in computer graphics that uses a small number of interval control points) inherited from the PIES, there are also some disadvantages. The main is connected with dense non-symmetric coefficient matrices and Gaussian elimination applied to solve the final system of algebraic equations. Unfortunately, the application of interval arithmetic and interval numbers also significantly slows the computational speed and utilizes more memory (RAM) than in the PIES. Usually, to accelerate computations, parallel computing methods (e.g. MPI or OpenMP) and graphics processing unit (GPU) for numerical calculations (such as CUDA or OpenCL) are commonly used. In our previous papers, we also proposed parallelization of the PIES by OpenMP [9] and CUDA [10] to reduce the time of computations. However, the use of these methods did not affect reducing RAM consumption. Therefore, to solve complex (large-scale) uncertainly defined problems using a standard personal computer (PC), we had to apply the fast multipole method (FMM) [11] to the IPIES in a similar way as in the FPIES. The FMM allows to significant reduction the RAM utilization [12]. It also reduces computation time.

The main goal of this paper is to present the IFPIES applied for numerical solving of 2D potential complex BVPs with uncertainly defined boundary shapes and boundary conditions. Simultaneous consideration of both uncertainties in describing the domain becomes a comprehensive approach to solving practical BVPs. The efficiency and accuracy of the IFPIES are tested on the potential problems with curvilinear domains.

2 Modelling uncertainties in the IFPIES

In previous papers (e.g. [2]), we described some problems during the application of either classical [13] or directed [14] interval arithmetic for modelling boundary problems with uncertainties. Hence, we also proposed some modifications during the application of the directed interval arithmetic, such as mapping arithmetic operators to the positive semi-axis into the IPIES (clearly described in [2]). The same strategy was applied in the IFPIES.

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The general form of the IFPIES formula was presented in [1], has the following form:

$$\frac{1}{2}u_l(\widehat{s}) = \sum_{j=1}^{n} \mathbb{R}\left\{ \int\limits_{s_{j-1}}^{s_j} \widehat{\boldsymbol{U}}_{lj}^{*(c)}(\widehat{s}, s)\boldsymbol{p}_j(s)\boldsymbol{J}_j^{(c)}(s)ds \right\} - \sum_{j=1}^{n} \mathbb{R}\left\{ \int\limits_{s_{j-1}}^{s_j} \widehat{\boldsymbol{P}}_{lj}^{*(c)}(\widehat{s}, s)\boldsymbol{u}_j(s)\boldsymbol{J}_j^{(c)}(s)ds \right\}, \tag{1}$$
$$l = 1, 2, ..., n, \ s_{l-1} \le \widehat{s} \le s_l, \ s_{j-1} \le s \le s_j,$$

where: $\widehat{s}$ and s are defined in the parametric coordinate system (as real values), s_{j-1} (s_{l-1}) correspond to the beginning while s_j (s_l) to the end of interval segment $\boldsymbol{S}_j$ ($\boldsymbol{S}_l$), n is the number of parametric segments that creates a boundary of the domain in 2D, $\widehat{\boldsymbol{U}}_{lj}^{*(c)}(\widehat{s}, s)$ and $\widehat{\boldsymbol{P}}_{lj}^{*(c)}(\widehat{s}, s)$ are modified interval kernels (complex function), $\boldsymbol{J}_j^{(c)}(s)$ is the interval Jacobian, $\boldsymbol{u}_j(s)$ and $\boldsymbol{p}_j(s)$ are interval parametric boundary functions on individual segments $\boldsymbol{S}_j$ of the interval boundary, $\mathbb{R}$ is the real part of complex function.

In this paper, for modelling uncertainly defined boundary shapes, curvilinear segments in the form of interval Bézier curves of the third degree are used:

$$\boldsymbol{S}_j(s) = \boldsymbol{a}_j s^3 + \boldsymbol{b}_j s^2 + \boldsymbol{c}_j s + \boldsymbol{d}_j, \quad 0 \le s \le 1, \tag{2}$$

where vector $\boldsymbol{S}_j(s) = [\boldsymbol{S}_j^{(1)}(s), \boldsymbol{S}_j^{(2)}(s)]^T$ is composed of two interval components connected with the direction of coordinates in 2D Cartesian reference system: $\boldsymbol{S}_j^{(1)} = [\underline{S}_j^{(1)}, \overline{S}_j^{(1)}], \boldsymbol{S}_j^{(2)} = [\underline{S}_j^{(2)}, \overline{S}_j^{(2)}]$. The $j = \{1, 2, ..., n\}$ is the number of segment created boundary, and s is a variable in the parametric reference system. Coefficients $\boldsymbol{a}_j, \boldsymbol{b}_j, \boldsymbol{c}_j, \boldsymbol{d}_j$ have also form of vectors composed of two intervals (similarly to $\boldsymbol{S}_j(s)$). They are computed using interval points describing particular segments of the boundary as presented in Fig. 1:

$$\boldsymbol{a}_j = \boldsymbol{P}_{e(j)} - 3\boldsymbol{P}_{i2(j)} + 3\boldsymbol{P}_{i1(j)} - \boldsymbol{P}_{b(j)}, \quad \boldsymbol{b}_j = 3(\boldsymbol{P}_{i2(j)} - 2\boldsymbol{P}_{i1(j)} + \boldsymbol{P}_{b(j)}),$$
$$\boldsymbol{c}_j = 3(\boldsymbol{P}_{i1(j)} - \boldsymbol{P}_{b(j)}), \quad \boldsymbol{d}_j = \boldsymbol{P}_{b(j)},$$

where coordinates of all points $\boldsymbol{P}$, regardless of their subscript, have the form of a vector of intervals:

$$\boldsymbol{P} = [\boldsymbol{P}^{(1)}, \boldsymbol{P}^{(2)}]^T = \left[[\underline{P}^{(1)}, \overline{P}^{(1)}], [\underline{P}^{(2)}, \overline{P}^{(2)}]\right]^T.$$

Boundary conditions are uncertainly defined using interval boundary functions $\boldsymbol{u}_j(s)$ and $\boldsymbol{p}_j(s)$ which are approximated by the following series:

$$\boldsymbol{u}_j(s) = \sum_{k=0}^{N} \boldsymbol{u}_j^{(k)} L_j^{(k)}(s), \quad \boldsymbol{p}_j(s) = \sum_{k=0}^{N} \boldsymbol{p}_j^{(k)} L_j^{(k)}(s), \tag{3}$$

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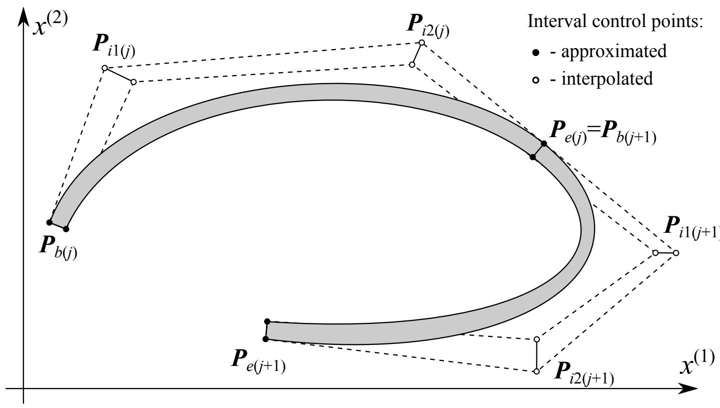

Fig. 1. The interval Bézier curve of the third degree used to define a segment of the boundary

where $\boldsymbol{u}_j^{(k)} = [\underline{u}_j^{(k)}, \overline{u}_j^{(k)}]$ and $\boldsymbol{p}_j^{(k)} = [\underline{p}_j^{(k)}, \overline{p}_j^{(k)}]$ are unknown or given interval values of boundary functions in defined points of the segment j, N - is the number of terms in approximating series (4), which approximated boundary functions on the segment j and $L_j^{(k)}(s)$ – the base functions (Lagrange polynomials) on segment j.

3 Solving the IFPIES

The process of solving the IFPIES is connected with the application of the FMM into the PIES. The FMM uses the tree structure to transform interactions between segments describing boundary into interactions between the cells (groups of segments). Also, the Taylor expansion is used to approximate the PIES's modified kernels. The process of applying the FMM into the PIES is clearly described in [3].

At last, integrals in (1) have the following form [1]:

$$\begin{gathered}
\int_{s_{j-1}}^{s_j} \widehat{\boldsymbol{U}}_{lj}^{*(c)}(\widehat{s}, s)\, p_j(s)\, \boldsymbol{J}_j^{(c)}(s) ds = \frac{1}{2\pi} \sum_{l=0}^{N_T} (-1)^l \cdot \\
\left\{ \sum_{k=0}^{N_T} \sum_{m=l}^{N_T} \frac{(k+m-1)! \cdot \boldsymbol{M}_k(\boldsymbol{\tau}_c)}{(\boldsymbol{\tau}_{el} - \boldsymbol{\tau}_c)^{k+m}} \cdot \frac{(\boldsymbol{\tau}'_{el} - \boldsymbol{\tau}_{el})^{m-l}}{(m-l)!} \right\} \frac{(\widehat{\boldsymbol{\tau}} - \boldsymbol{\tau}'_{el})^l}{l!}, \\
\int_{s_{j-1}}^{s_j} \widehat{\boldsymbol{P}}_{lj}^{*(c)}(\widehat{s}, s)\, u_j(s)\, \boldsymbol{J}_j^{(c)}(s) ds = \frac{1}{2\pi} \sum_{l=0}^{N_T} (-1)^l \cdot \\
\left\{ \sum_{k=1}^{N_T} \sum_{m=l}^{N_T} \frac{(k+m-1)! \cdot \boldsymbol{N}_k(\boldsymbol{\tau}_c)}{(\boldsymbol{\tau}_{el} - \boldsymbol{\tau}_c)^{k+m}} \cdot \frac{(\boldsymbol{\tau}'_{el} - \boldsymbol{\tau}_{el})^{m-l}}{(m-l)!} \right\} \frac{(\widehat{\boldsymbol{\tau}} - \boldsymbol{\tau}'_{el})^l}{l!}.
\end{gathered} \tag{4}$$

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where: N_T is the number of terms in the Taylor expansion, $\widehat{\boldsymbol{\tau}} = \boldsymbol{S}_l^{(1)}(\widehat{s}) + i\boldsymbol{S}_l^{(2)}(\widehat{s})$, $\boldsymbol{\tau} = \boldsymbol{S}_j^{(1)}(s) + i\boldsymbol{S}_j^{(2)}(s)$, complex interval points $\boldsymbol{\tau}_c$, $\boldsymbol{\tau}_{el}$, $\boldsymbol{\tau}_c'$, $\boldsymbol{\tau}_{el}'$ are midpoints of leaves obtained while tracing the tree structure (see [15]). Expressions $\boldsymbol{M}_k(\boldsymbol{\tau}_c)$ and $\boldsymbol{N}_k(\boldsymbol{\tau}_c)$ are called moments (and they are computed twice only) and have the form [1]:

$$\begin{aligned} \boldsymbol{M}_k(\boldsymbol{\tau}_c) &= \int\limits_{s_{j-1}}^{s_j} \frac{(\boldsymbol{\tau} - \boldsymbol{\tau}_c)^k}{k!} p_j(s) \boldsymbol{J}_j^{(c)}(s) ds, \\ \boldsymbol{N}_k(\boldsymbol{\tau}_c) &= \int\limits_{s_{j-1}}^{s_j} \frac{(\boldsymbol{\tau} - \boldsymbol{\tau}_c)^{k-1}}{(k-1)!} \boldsymbol{n}_j^{(c)} u_j(s) \boldsymbol{J}_j^{(c)}(s) ds. \end{aligned} \tag{5}$$

where $\boldsymbol{n}_j^{(c)} = \boldsymbol{n}_j^{(1)} + i\boldsymbol{n}_j^{(2)}$ the complex interval normal vector to the curve created segment j.

The IFPIES, similarly to the original PIES, is written at collocation points whose number corresponds to the number of unknowns. However, in the IFPIES, the system of algebraic equations $\boldsymbol{A} \cdot \boldsymbol{x} = \boldsymbol{b}$ is produced implicitly, i.e. only the result of multiplication of the matrix $\boldsymbol{A}$ by the vector of unknowns $\boldsymbol{x}$ is obtained, contrary to the explicit form in the PIES. Therefore, an iterative GMRES solver [16] modified by the application of directed interval arithmetic directly integrated with the FMM was applied in the IFPIES. Also, the GMRES solver was applied to the IPIES to obtain a more reliable comparison.

4 Numerical results

The example is the gear-shaped plate presented in Fig. 2. The problem is described by Laplace's equation. The boundary contains 2 048 curvilinear interval segments. Interval boundary conditions are also presented in Fig. 2 (where u - Dirichlet and p - Neumann boundary conditions). Tests are performed on a PC based on Intel Core i5-4590S with 16 GB RAM. Application of the IPIES and the IFPIES are compiled by g++ 7.5.0 (-O2 optimization) on 64-bit Ubuntu Linux OS (kernel 5.4.0).

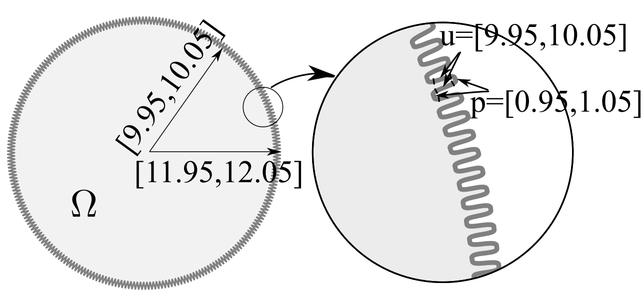

Fig. 2. Considered the gear-shaped modelled by curvilinear segments

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The first research focused on finding the optimal number of tree levels in the IFPIES from the speed of computations and RAM utilization point of view. Approximation of the IFPIES kernels uses 25 terms in the Taylor series, and the GMRES tolerance is equal to 10^{-8}.

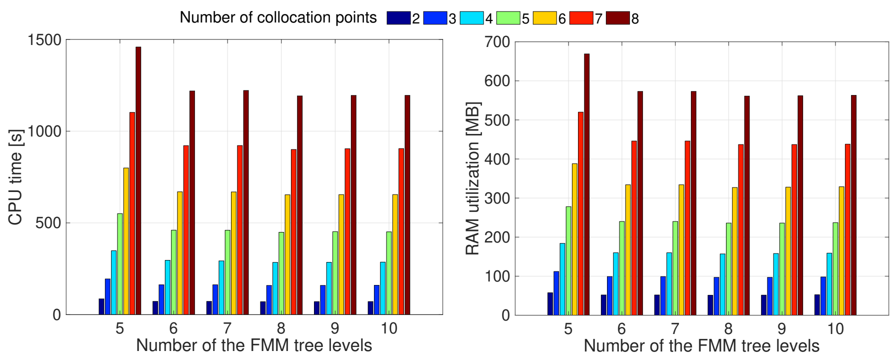

Fig. 3. Comparison of computation time and RAM utilization of the IFPIES for different tree levels

As can be seen from Fig. 3, the shortest time of computations and the smallest number of utilized memory for all numbers of collocation points is obtained for 8 tree levels. Therefore, that number is used in other research.

The subsequent research focused on the CPU time, RAM utilization and accuracy of the IFPIES compared to the IPIES only due to the lack of methods of solving problems with uncertainly defined boundary shape and boundary conditions. The same number of terms in the Taylor series and the value of GMRES tolerance as in the previous example are used. The number of collocation points is the same in each segment, which is changed from 2 to 8. Therefore, we should solve the system of 4 096 to 16 384 equations, respectively.

Table 1. Comparison between the IFPIES and the IPIES

Number of		CPU time [s]		RAM utilization [MB]		MSE	
col. pts	eqs	*IFPIES*	*IPIES*	*IFPIES*	*IPIES*	inf	sup
2	4 096	70.73	131.24	51.31	390	0.0	0.0
3	6 144	159.22	310.25	97	896	$4.41 \cdot 10^{-15}$	$1.43 \cdot 10^{-15}$
4	8 192	284.84	565.46	157	1 578	$6.29 \cdot 10^{-15}$	$5.96 \cdot 10^{-15}$
5	10 240	449.05	910.28	236	2 455	$6.23 \cdot 10^{-14}$	$2.55 \cdot 10^{-14}$
6	12 288	653.36	1 349.4	327	3 523	$1.17 \cdot 10^{-11}$	$2.39 \cdot 10^{-13}$
7	14 336	900.19	1 892.83	437	4 787	$3.33 \cdot 10^{-13}$	$6.74 \cdot 10^{-11}$
8	16 384	1 192.09	2 559.22	561	6 243	$6.95 \cdot 10^{-10}$	$6.14 \cdot 10^{-11}$

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As can be seen from Tab. 1, the IFPIES is about 2 times faster and uses up to 10 times less RAM than the IPIES. To prove the accuracy of the proposed method, the mean square error (MSE) between the lower and upper bound (infimum and supremum) of the IFPIES and the IPIES solutions are computed. The IFPIES is as accurate as the IPIES. The mean square error (MSE) between both methods is very low and does not exceed 10^{-9}. Hence, the IFPIES is as accurate as the IPIES.

5 Conclusions

The paper presents the IFPIES in solving 2D potential curvilinear uncertainly defined boundary value problems. The IFPIES was previously applied in modelling and solving 2D polygonal potential problems with the uncertainly defined shape of the boundary. Applied interval modelling of boundary shape and boundary conditions allows for including the uncertainty of measurement data (measurement errors) in calculations, which is ignored in classic practical design. Also, applying the fast multipole technique in the IFPIES allows for the highly efficient solving of complex engineering problems on a standard PC in a reasonable time. However, the real power of the IFPIES is connected with low RAM utilization. The IPIES for solving the problems with a system of 16 384 equations uses over 6 GB RAM, while the IFPIES requires about 0.5 GB of RAM.

Obtained results suggest that the direction of research should be continued. Our further research should cover problems modelled by other than Laplace's equations.

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