Acceleration of Interval PIES Computations Using Interpolation of Kernels with Uncertainly Defined Boundary Shape

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Abstract. In this paper, interval modification of degenerate parametric integral equation system (DPIES) for Laplace's equation was presented. The main purpose of such a modification was to reduce the computational time necessary to solve uncertainly defined boundary problems. Such problems were modeled with the uncertainly defined shape of the boundary using proposed modified directed interval arithmetic. The presented algorithm of mentioned interval modification of DPIES was implemented as a computer program. The reliability and efficiency tests were proceeded based on boundary problems modeled by Laplace's equation. Both, obtained solutions, as well as computational time, were analyzed. As a result, the interval kernels interpolation (using Lagrange polynomial) caused a reduction of necessary interval arithmetic calculations, which also caused accelerations of computations.

Keywords: boundary value problems · collocation method · parametric integral equation system · interval arithmetic · uncertainty.

1 Introduction

The finite element method (FEM) [1] and boundary element method (BEM) [2] are the most often used methods for solving boundary problems. Nowadays, it is very important to include the uncertainty of measurements in solving such problems. Therefore, a lot of modifications have appeared in the literature, such as stochastic [3], interval [4] or fuzzy [5]. However, the disadvantages of FEM and BEM, mainly the necessity of discretization, caused an increasing amount of calculations and extend the computation time.

Therefore, the parametric integral equation system (PIES) [6,7] (where the classical discretization was eliminated) was proposed. The advantages of this method in solving uncertainly defined problems were presented in [8,9]. These papers focused on the simplest and often used interval arithmetic [10,11] for modeling the uncertainty. Unfortunately, the direct application was time-consuming, and obtained solutions were often overestimated.

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Results of degenerate PIES (DPIES) application for solving boundary problems [12] was the motivation to test its applicability for uncertainly defined problems. The mathematical formalism of DPIES allowed the development of more efficient algorithms for the numerical solution of PIES.

In this paper, interval modification of DPIES (to include the uncertainty) was presented. Uncertainty modeling is more reliable because it includes e.g. measurement errors. The proposed strategy was implemented as a computer program and tested on examples of problems described by Laplace's equation. Interval DPIES occurs to be faster than the interval PIES [8,9,6]. The additional advantage was the reduction of the number of interval calculations. This reduced not only the computational time but also the unnecessary overestimations.

2 IPIES with Boundary Shape Uncertainty

The application of interval arithmetic [10,11], to model the boundary shape uncertainty, seemed to be sensible and elementary. However, obtained solutions to so-defined boundary problems raised doubts about their reliability and usefulness. Details were discussed in [9] and the interval PIES (IPIES) [8,9] using a modification of the directed interval arithmetic was proposed. The solutions on the boundary (of two-dimensional problems modeled by Laplace's equation with boundary shape uncertainty) can be obtained by solving IPIES [9]:

$$0.5u_l(z) = \sum_{j=1}^{n} \int_{\widehat{s}_{j-1}}^{\widehat{s}_j} \{\boldsymbol{U}^*_{lj}(z,s)p_j(s) - \boldsymbol{P}^*_{lj}(z,s)u_j(s)\}\boldsymbol{J}_j(s)ds, \tag{1}$$

where $\widehat{s}_{l-1} \leq z \leq \widehat{s}_l$ and $\widehat{s}_{j-1} \leq s \leq \widehat{s}_j$ are exactly defined in parametric coordinate system. They correspond to beggining and ending of segment of interval curve $S_m = [\boldsymbol{S}^{(1)}_m, \boldsymbol{S}^{(2)}_m]$ $(m = j, l)$ which model uncertainly defined boundary shape. The function $\boldsymbol{J}_j(s) = [\underline{J}_j(s), \overline{J}_j(s)]$ is the Jacobian of interval curve segment $\boldsymbol{S}_j(s)$. Functions $\boldsymbol{U}^*_{lj}(z,s) = [\underline{U}^*_{lj}(z,s), \overline{U}^*_{lj}(z,s)], \boldsymbol{P}^*_{lj}(z,s) = [\underline{P}^*_{lj}(z,s), \overline{P}^*_{lj}(z,s)]$ are interval kernels defined as:

$$\boldsymbol{U}^*_{lj}(z,s) = \frac{1}{2\pi} ln\left(\frac{1}{[\boldsymbol{\eta}_1^2 + \boldsymbol{\eta}_2^2]^{0.5}}\right), \quad \boldsymbol{P}^*_{lj}(z,s) = \frac{1}{2\pi}\frac{\boldsymbol{\eta}_1\boldsymbol{n}_1(s) + \boldsymbol{\eta}_2\boldsymbol{n}_2(s)}{\boldsymbol{\eta}_1^2 + \boldsymbol{\eta}_2^2}, \tag{2}$$

where $\boldsymbol{n}_1(s) = [\underline{n}_1(s), \overline{n}_1(s)], \boldsymbol{n}_2(s) = [\underline{n}_2(s), \overline{n}_1(s)]$ are interval components of normal vector $n = [n_1(s), n_2(s)]^T$ to interval segment $\boldsymbol{S}_j$. Kernels include the boundary shape uncertainty by the relationship between interval segments $S_m(m = l, j = 1, 2, 3, ..., n)$ (defined in the Cartesian coordinate system):

$$\boldsymbol{\eta}_1 = \boldsymbol{S}^{(1)}_l(z) - \boldsymbol{S}^{(1)}_j(s), \quad \boldsymbol{\eta}_2 = \boldsymbol{S}^{(2)}_l(z) - \boldsymbol{S}^{(2)}_j(s), \tag{3}$$

where $S_m(m = j, l)$ are segments of interval closed curves.

Integral functions $p_j(s), u_j(s)$ are parametric boundary functions defined on segments $\boldsymbol{S}_j$. One of these functions is defined by boundary conditions, then the other is searched by numerical solution of IPIES. The boundary conditions were defined exactly (without uncertainty) to unambiguously analyze the influence of interval kernels interpolation.

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3 Interval Degenerated Kernels

The direct application of the collocation and Galerkin methods to solve PIES was presented in [7,13]. The collocation method was easily applied but produced less accurate solutions. The Galerkin method was more accurate but needed more computational time. Therefore, in this work, to accelerate the calculations, the interval kernels in IPIES were replaced by degenerate ones. The literature provides methods (without the uncertainty), such as Fourier or Taylor series expansion [14,15] to obtain such kernels. However, in this paper, a new strategy using Lagrange polynomials was proposed.

3.1 Generalized Lagrange Polynomials

The form (1) was used for $l = j$ because of the kernels singularity. Interpolation was used only beyond the main diagonal ($j \neq l$). The application of generalized Lagrange interpolation (for functions of two variables) was very simple, because the kernels $\boldsymbol{X}_{lj}^{*}(z,s)(\boldsymbol{X} = \boldsymbol{U}, \boldsymbol{P})$ already were defined in a unit square using normalized parameters $0 \leq z, s \leq 1$:

$$\boldsymbol{X}_{lj}^{*}(z,s) = \sum_{a=0}^{p-1} \sum_{b=0}^{m-1} \boldsymbol{X}_{lj}^{(ab)} L_l^{(a)}(z) L_j^{(b)}(s), \tag{4}$$

$$L_l^{(a)}(z) = \prod_{k=0,k\neq a}^{p-1} \frac{z - z^{(k)}}{z^{(a)} - z^{(k)}}, \quad L_j^{(b)}(s) = \prod_{k=0,k\neq b}^{m-1} \frac{s - s^{(k)}}{s^{(b)} - s^{(k)}}, \tag{5}$$

and interval values of $\boldsymbol{X}_{lj}^{(ab)} = \boldsymbol{X}_{lj}(z^{(a)}, s^{(b)})$ were easily determined from the formulas (2) at the interpolation points $z^{(a)}, s^{(b)}$. These points were defined by roots of Chebyshev polynomials, to avoid Runge's phenomenon. The number of such nodes and their distribution determine the interpolation accuracy. Only $\boldsymbol{X}_{lj}^{(ab)}$ was defined as intervals, so the amount of interval data was significantly reduced. Substituting (4) to the (1) the degenerate interval parametric integral equation system (DIPIES) (for $l \neq j$) was obtained:

$$\begin{aligned} 0.5u_l(z) = \sum_{j=1}^{n} \Bigg\{ & \sum_{a=0}^{p-1} \sum_{b=0}^{m-1} \boldsymbol{U}_{lj}^{(ab)} L_l^{(a)}(z) \int_0^1 L_j^{(b)}(s) p_j(s) \\ & - \sum_{a=0}^{p-1} \sum_{b=0}^{m-1} \boldsymbol{P}_{lj}^{(ab)} L_l^{(a)}(z) \int_0^1 L_j^{(b)}(s) u_j(s) \Bigg\} \boldsymbol{J}_j(s) ds, \end{aligned} \tag{6}$$

where $l = 1, 2, ..., n$ and for $l = j$ general PIES (1) was used.

The separation of the variables made it possible to move the Lagrange polynomials $L_l^{(a)}(z)$ and interval values $\boldsymbol{P}_{lj}^{(ab)}, \boldsymbol{U}_{lj}^{(ab)}$ outside the integral. The uncertainty of the integrand is determined only by Jakobian $\boldsymbol{J}_j(s)$. Other values (polynomials $L_j^{(b)}(s)$ and unknown functions $u_j(s)$ or $p_j(s)$) are exactly defined (without uncertainty).

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3.2 Numerical Solution

Solution of (6) is to find unknown functions $p_j(s)$ or $u_j(s)$. They were approximated by $\widetilde{p}_j$ or $\widetilde{u}_j$ series using Chebyshev polynomials as base functions $f_j^{(k)}$:

$$\widetilde{p}_j(s) = \sum_{k=0}^{M-1} p_j^{(k)} f_j^{(k)}(s), \quad \widetilde{u}_j(s) = \sum_{k=0}^{M-1} u_j^{(k)} f_j^{(k)}(s), \qquad j = 1, ..., n. \tag{7}$$

The collocation method was applied to determine the unknown $u_j^{(k)}$ and $p_j^{(k)}$. In this method, the equation (6) is written in the so-called collocation points $z^{(c)}$, where $s_{l-1} < z^{(c)} < s_l$. Substituting approximating series (7) to (6) degenerate IPIES (for $l \neq j$) is presented as:

$$0.5 \sum_{k=0}^{M} u_l^{(k)} f_l^{(k)}(z^{(c)}) = \sum_{j=1}^{n} \left\{ \sum_{a=0}^{p-1} \sum_{b=0}^{m-1} \boldsymbol{U}_{lj}^{(ab)} L_l^{(a)}(z^{(c)}) \int_0^1 L_j^{(b)}(s) \sum_{k=0}^{M} p_j^{(k)} f_j^{(k)}(s) \right.$$
$$\left. - \sum_{a=0}^{p-1} \sum_{b=0}^{m-1} \boldsymbol{P}_{lj}^{(ab)} L_l^{(a)}(z^{(c)}) \int_0^1 L_j^{(b)}(s) \sum_{k=0}^{M} u_j^{(k)} f_j^{(k)}(s) \right\} \boldsymbol{J}_j(s) ds, \tag{8}$$

where $l = 1, 2, ..., n$ and for $l = j$ the series (7) were used in IPIES (1).

The DIPIES presented at the collocation points $z^{(c)}$ can be obtained (in an explicit form) for any boundary problem. The values obtained by multiplying the Lagrange polynomial $L_l^{(a)}(z^{(c)})$ by the interval values $\boldsymbol{U}_{lj}^{(ab)}$ or $\boldsymbol{P}_{lj}^{(ab)}$ create a vertical vector, whereas, the values obtained by calculating the integrals create a horizontal vector. Therefore, using k collocation points on all n segments, a two $n \times k$ - dimensional vectors are obtained.

Even theoretically, such a strategy has a significant advantage over IPIES. Till now, each element of the matrix required computing integrals with interval kernels. In proposed DIPIES, only the elements on the diagonal are calculated directly from (1). Outside the main diagonal, they are calculated as the product of two previously obtained vectors. This is a much less time-consuming operation than interval integration. Additionally, the number of interval arithmetic calculations (necessary to obtain the matrix coefficients) was significantly reduced.

4 Tests of the Proposed Strategy

The solutions' obtained using proposed DIPES are compared with those obtained using IPIES. A different number of interpolation nodes (for the $\boldsymbol{U}_{lj}$ and $\boldsymbol{P}_{lj}$ kernels) was considered. Intel Core i5-4590S with 8 GB RAM with MS Visual Studio 2013 (version: 12.0.21005.1 REL) compiler on Windows 8.1 64-bit system was used during tests. Although the average time from 100 runs of the algorithm was presented, the again obtained values (even for the same nodes number) can differ slightly. The exact interpolation of kernel P for adjacent segments occurred to be troublesome, so it was also obtained by classical integration (1). The examples were solved using three collocation points on each segment.

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4.1 Elementary Example with an Analytical Solution

The elementary problem (presented in Fig. 1) was considered to confirm the correctness of the obtained solutions. The exactly defined analytical solution is $u = 100y$. Table 1 presents chosen solutions and the average relative error between IPIES and DIPIES solutions (calculated separately for the lower and upper bound of the interval). The average time necessary to calculate the elements of the G matrix (U kernel) and H matrix (P kernel) are also presented. All solutions were obtained at 20 points in the cross-section, where $x = 0.75$ and y changes from 0 to 1 (dashed line in Fig. 1).

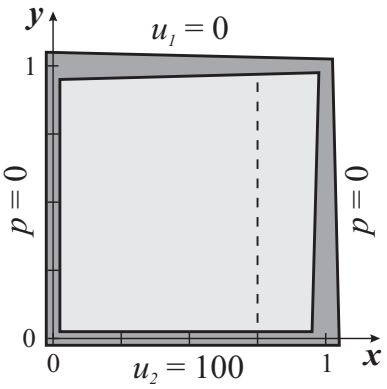

Fig. 1. The problem example with shape uncertainty.

Table 1. Solutions in the domain (computational time and the average relative error).

$100y$	IPIES	DIPIES ($U \times P$)			
		3x2	2x3	3x3	3x5
20	[18.87, 20.92]	[19.07, 21.16]	[18.89, 21.12]	[18.82, 20.89]	[18.83, 20.89]
40	[40.36, 39.70]	[40.27, 39.66]	[40.25, 39.81]	[40.34, 39.70]	[40.34, 39.69]
60	[61.60, 58.65]	[61.44, 58.52]	[61.50, 58.71]	[61.62, 58.67]	[61.62, 58.67]
80	[82.55, 77.78]	[82.46, 77.70]	[82.37, 77.70]	[82.58, 77.82]	[82.58, 77.82]
average relative error [%]		[1.65, 0.98]	[1.02, 0.80]	[0.13, 0.07]	[0.08, 0.06]
time U [ms]	10.91	3.11	2.85	3.19	3.43
time P [ms]	6.64	4.76	4.93	5.17	5.86

The number of interpolation nodes was defined as $U \times P$. For example, 3×2 means the 3 nodes for U kernel and 2 for P kernel. Exact analytical solutions are located inside all of the interval solutions. Despite the elementary example, the DIPIES method occurs to be much faster than IPIES (especially for the U kernel) with the average relative error lower than 0.1%.

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4.2 Example of a Problem with a Complex Shape (40 Segments)

The shape of the boundary in the next example was defined using 40 segments. The considered uncertainly defined shape with the boundary conditions is presented in Fig. 2. Solutions of such a problem were obtained in the cross-section (dashed line in Fig. 2), where $y = 3$ and x changes from 1 to 20.

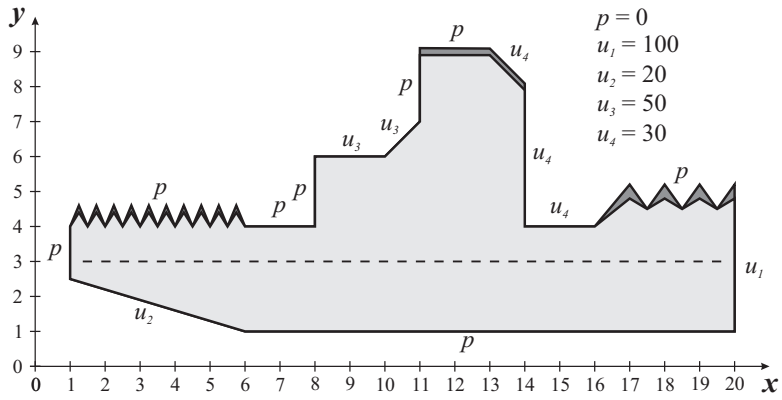

Fig. 2. The problem example with shape uncertainty.

The computational time and the average relative error of the solutions are presented in Tab. 2. Even three nodes are sufficient, for interpolation of U kernel, to obtain a solution with the error of 0.03%. Unfortunately, despite the exclusion of P kernel interpolation for adjacent segments, similar accuracy was obtained using eight nodes. However, even then, the DIPIES is faster than IPIES.

Table 2. Solutions in the domain (computational time and the average relative error).

	IPIES	DIPIES ($U \times P$)				
		2x8	3x3	3x5	3x8	5x8
time U [ms]	983.08	53.34	78.85	79.56	76.22	151.75
time P [ms]	625.23	491.2	125.78	226.27	502.51	459.12
average relative error [%]	[1.67, 1.68]	[0.64,0.91]	[0.29, 0.35]	[0.04, 0.03]	[0.05, 0.06]	

5 Conclusions

The paper presents DIPIES, obtained by replacing kernels in IPIES with degenerate ones. The generalized Lagrange interpolation was used to obtain such kernels. The accuracy of the interpolation is determined by the number of nodes and their arrangement. Moreover, for the interpolation of the U kernel, a smaller

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number of nodes was sufficient compared to the P kernel. Therefore, the P kernel interpolation was used only for not adjacent segments. The effectiveness of the strategy was tested on the example of problems (modeled by Laplace's equation) with an uncertainly defined boundary shape. Two examples were solved. The results were compared with the analytical and numerical solutions (obtained by IPIES). Obtained results, present a high potential of the method. The application of degenerate kernels in the IPIES reduced the number of interval arithmetic operations and accelerated the calculations with satisfactory solutions accuracy.

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