Derivation and Computation of Integro-Riccati Equation for Ergodic Control of Infinite-dimensional SDE

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Abstract. Optimal control of infinite-dimensional stochastic differential equations (SDEs) is a challenging topic. In this contribution, we consider a new control problem of an infinite-dimensional jump-driven SDE with long (sub-exponential) memory arising in river hydrology. We deal with the case where the dynamics follow a superposition of Ornstein–Uhlenbeck processes having distributed reversion speeds (called supOU process in short) as found in real problems. Our stochastic control problem is of an ergodic type to minimize a long-run linear-quadratic objective. We show that solving the control problem reduces to finding a solution to an integro-Riccati equation and that the optimal control is infinite-dimensional as well. The integro-Riccati equation is numerically computed by discretizing the phase space of the reversion speed. We use the supOU process with an actual data of river discharge in a mountainous river environment. Computational performance of the proposed numerical scheme is examined against different discretization parameters. The convergence of the scheme is then verified with a manufactured solution. Our paper thus serves as new modeling, computation, and application of an infinite-dimensional SDE.

Keywords: Infinite-dimensional Stochastic Differential Equation, Stochastic Control in Infinite-Dimension, Integro-Riccati equation

1. Introduction

Optimal control of stochastic partial differential equations, namely infinite-dimensional stochastic differential equations (SDEs), has recently been a hot research topic from both theoretical and engineering sides because of their rich mathematical properties and importance in applied problems [1]. Such examples include but are not limited to shape optimization under uncertainty [2], evolution theory of age-dependent population dynamics [3], and portfolio management [4].

The main difficulty in handling a control problem of an infinite-differential SDE comes from the infinite-dimensional nature of the optimality equation. Indeed, in a conventional control problem of a finite-dimensional SDE, finding an optimal control

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reduces to solving an optimality equation given as a finite-dimensional parabolic partial differential equation. Its solution procedure can be constructed by a basic numerical method like a finite difference scheme [5]. By contrast, a control problem of an infinite-differential SDE involves a partial differential equation having an infinite dimension, which cannot be solved numerically in general. A tailored numerical scheme is necessary to handle the infinite-dimensional nature [6, 7]; such schemes have not always been applied to problems with actual system dynamics. This issue is a bottleneck in applications of infinite-differential SDEs in engineering problems. Hence, demonstrating a computable example of interest in an engineering problem can be useful for better understanding the control of infinite-differential SDEs.

The objectives of this paper are to present an infinite-dimensional SDE arising in hydrology and environmental management, and to formulate its ergodic linear-quadratic (LQ) control problem. The system governs temporal evolution of river discharge as a superposition of Ornstein–Uhlenbeck processes (supOU process) as recently identified in Yoshioka [8]. Markovian stochastic modeling of river discharge has long been a standard method for assessing streamflows [9]. However, some researchers including the first author recently found that the Markovian assumption is often inappropriate for discharge time series of actual perennial river environments due to the sub-exponential auto-correlation [8]. This sub-exponential auto-correlation is consistent with the supOU process as an infinite-dimensional SDE, which is why we are focusing on this specific stochastic process.

The goal of our control problem is to modulate the discharge considering a water demand with a least effort in long-run. This can be the simplest management problem of water resources in which maintaining the water depth or discharge near some prescribed level is preferred. The LQ nature allows us to reduce the infinite-dimensional optimality equation to a two-dimensional integro-Riccati equation which is computable by a collocation method in space [10] combined with a forward Euler method in time. This integro-Riccati equation itself has not been derived in the literature so far. Focusing on an actual parameter set, we provide computational examples of the optimal control along with their well-posedness and optimality. Our problem is simple but involves several nontrivial scientific issues to be tackled in future. We believe that this contribution would advance modeling problems with uncertainty from a viewpoint of infinite-dimensional SDEs.

2. Control Problem

2.1 Uncontrolled System

We consider a control problem of discharge at a point in a river, which is a continuous-time and continuous-state scalar variable denoted as X_t at time $t \geq 0$ with an initial condition $X_0 \geq 0$. Our formulation is based on the SDE representation of supOU processes suggested in Barndorff-Nielsen [11] and later justified in Barndorff-Nielsen and Stelzer [12]. The assumptions made in our SDE are based on the physical consideration of river discharge as a jump-driven process [8].

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The system dynamics without control follow the distributed SDE

$$\mathrm{d}X_t = \underline{X} + \int_0^{+\infty} Y_t(\lambda)\pi(\mathrm{d}\lambda),\ t > 0 \tag{1}$$

with a minimum discharge $\underline{X} \geq 0$ and $Y_t(\cdot)$ ($t \in \mathbb{R}$) governed by

$$\mathrm{d}Y_t(\lambda) = -\lambda Y_t(\lambda)\mathrm{d}t + \mathrm{d}L_t(\lambda),\ t > 0. \tag{2}$$

Here, π is a probability measure of a positive random variable absolutely continuous with respect to $\mathrm{d}\lambda$ on the half line $\lambda > 0$, such that

$$\int_0^{+\infty} \frac{\pi(\mathrm{d}\lambda)}{\lambda} < +\infty, \tag{3}$$

and $L_t(\cdot)$ ($t > 0$) is a pure positive-jump space-time Lévy process corresponding to an ambit field whose background Lévy measure $v = v(\mathrm{d}z)$ is a finite-variation type:

$$\int_0^{+\infty} \min\{1, z\} v(\mathrm{d}z),\ \int_0^{+\infty} z^2 v(\mathrm{d}z) < +\infty. \tag{4}$$

The conditions (3) and (4) are imposed to well-define jumps of the SDE (2) and to guarantee boundedness of the statistical moments of discharge [8]. The expectation of $\mathrm{d}L_t(\lambda)\mathrm{d}L_t(\theta)$ ($t > 0$, $\lambda, \theta > 0$) is formally given by

$$\mathbb{E}\left[\mathrm{d}L_s(\lambda)\mathrm{d}L_s(\theta)\right] = \begin{cases} M_1^2(\mathrm{d}s)^2 & (\theta \neq \lambda) \\ M_2 \delta_{\{\theta=\lambda\}} \mathrm{d}\lambda / \pi(\mathrm{d}\lambda)\mathrm{d}s & (\theta = \lambda) \end{cases},\ M_k = \int_0^{+\infty} z^k v(\mathrm{d}z)\ (k = 1, 2) \tag{5}$$

with the Dirac delta δ. The noise process associated with the supOU process therefore is not of a trace class [13], suggesting that the system dynamics are highly irregular. This point will be discussed in the next section.

The SDE representation (1) implies that the river discharge is multi-scale in time because it is a superposition of infinitely many independent OU processes having different reversion speeds λ on the probability measure π (i.e., different values of the decay speed of flood pulses). More specifically, in the supOU process, each jump of X associates a corresponding λ generated from π [11], allowing for the existence of flood pulses decaying with different speeds. In principle, this kind of multi-scale nature cannot be reproduced by simply using a classical OU process because it has only one decay speed. The supOU processes are therefore expected to be a more versatile alternative to the classical OU ones. The parameters of the densities π and v were successfully identified in Yoshioka [8], which will be used later.

2.2 Controlled System

The controlled system is the SDE (1) with Y now governed by

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$$\mathrm{d}Y_t(\lambda) = \left(-\lambda Y_t(\lambda) + u_t(\lambda)\right)\mathrm{d}t + \mathrm{d}L_t(\lambda),\ t > 0. \tag{6}$$

Here, u is a control variable progressively measurable with respect to a natural filtration generated by $L_t(\cdot)$ ($t > 0$), and satisfies the square integrability conditions

$$\limsup_{T\to+\infty} \frac{1}{T}\mathbb{E}\left[\int_0^T \int_0^{+\infty} \left(u_s(\lambda)\right)^2 \pi(\mathrm{d}\lambda)\mathrm{d}s\right],\ \limsup_{T\to+\infty} \frac{1}{T}\mathbb{E}\left[\int_0^T X_s^2 \mathrm{d}s\right] < +\infty. \tag{7}$$

Our objective functional is the following long-run LQ type:

$$J(u) = \limsup_{T\to+\infty} \frac{1}{T}\mathbb{E}\left[\frac{1}{2}\int_0^T \left\{\left(X_s - \bar{X}\right)^2 + w\int_0^{+\infty} \left(u_s(\lambda)\right)^2 \pi(\mathrm{d}\lambda)\right\}\mathrm{d}s\right] \tag{8}$$

with a target discharge $\bar{X} > 0$ representing a water demand and $w > 0$ is a weight balancing the two terms: the deviation from the target and control cost. The objective is to find the minimizer $u = u^*$ of $J(u)$: $H = \inf_u J(u) \geq 0$. Note that the jumps are not controlled as they represent uncontrollable inflow events from upstream.

2.3 Integro-Riccati Equation

By a dynamic programming argument [e.g., 14], we infer that the optimality equation of the control problem is the infinite-dimensional integro-partial differential equation

$$\begin{aligned}
&-H + \inf_{u(\cdot)}\left\{\int_0^{+\infty}\left(-\lambda Y(\lambda) + u(\lambda)\right)\nabla V(Y)\pi(\mathrm{d}\lambda) + \frac{w}{2}\int_0^{+\infty}\left(u(\lambda)\right)^2 \pi(\mathrm{d}\lambda)\right\} \\
&+\int_0^{+\infty}\left(\int_0^{+\infty} V\left(Y(\cdot) + z\delta_{\{\omega=(\cdot)\}}\frac{\mathrm{d}(\cdot)}{\pi(\mathrm{d}(\cdot))}\right)\pi(\mathrm{d}\omega) - V(Y)\right)\nu(\mathrm{d}z) \qquad , Y \in L^2(\pi), \\
&+\frac{1}{2}\int_0^{+\infty}\int_0^{+\infty} Y(\lambda)Y(\theta)\pi(\mathrm{d}\lambda)\pi(\mathrm{d}\theta) - \bar{X}\int_0^{+\infty} Y(\lambda)\pi(\mathrm{d}\lambda) + \frac{1}{2}\bar{X}^2 = 0
\end{aligned} \tag{9}$$

where $L^2(\pi)$ is a collection of square integrable functions with respect to π, and $\nabla V(Y)$ is the Fréchet derivative identified as a mapping from $L^2(\pi)$ to $L^2(\pi)$. The infimum in (9) must be taken with respect to functions belonging to $L^2(\pi)$. By calculating the inf term, (9) is rewritten as

$$\begin{aligned}
&-H - \int_0^{+\infty} \lambda \nabla V(Y) Y(\lambda)\pi(\mathrm{d}\lambda) - \frac{1}{2w}\int_0^{+\infty}\left(\nabla V(Y)\right)^2 \pi(\mathrm{d}\lambda) \\
&+\int_0^{+\infty}\left(\int_0^{+\infty} V\left(Y(\cdot) + z\delta_{\{\omega=(\cdot)\}}\frac{\mathrm{d}(\cdot)}{\pi(\mathrm{d}(\cdot))}\right)\pi(\mathrm{d}\omega) - V(Y)\right)\nu(\mathrm{d}z) \qquad , Y \in L^2(\pi) \\
&+\frac{1}{2}\int_0^{+\infty}\int_0^{+\infty} Y(\lambda)Y(\theta)\pi(\mathrm{d}\lambda)\pi(\mathrm{d}\theta) - \bar{X}\int_0^{+\infty} Y(\lambda)\pi(\mathrm{d}\lambda) + \frac{1}{2}\bar{X}^2 = 0
\end{aligned} \tag{10}$$

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with (a candidate of) the optimal control as a minimizer of the infimum in the first line of (9):

$$u^*(Y) = -\frac{1}{w}\nabla V(Y),\ Y \in L^2(\pi). \tag{11}$$

A formal solution to (9) is a couple (V,h) of smooth $V : L^2(\pi) \to \mathbb{R}$ and $h \in \mathbb{R}$.

Invoking the LQ nature of our problem suggests the ansatz: for any $Y \in L^2(\pi)$,

$$V(Y) = \frac{1}{2}\int_0^{+\infty}\int_0^{+\infty}\Gamma(\theta,\lambda)Y(\lambda)Y(\theta)\pi(\mathrm{d}\lambda)\pi(\mathrm{d}\theta) + \int_0^{+\infty}\gamma(\lambda)Y(\lambda)\pi(\mathrm{d}\lambda) \tag{12}$$

with symmetric $\pi\otimes\pi$-integrable Γ and π-integrable γ. Substituting (12) into (10) yields our integro-Riccati equation

$$0 = -(\theta+\lambda)\Gamma(\theta,\lambda) - \frac{1}{w}\int_0^{+\infty}\Gamma(\theta,\omega)\Gamma(\omega,\lambda)\pi(\mathrm{d}\omega) + 1,\ \lambda,\theta > 0, \tag{13}$$

$$0 = -\lambda\gamma(\lambda) - \frac{1}{w}\int_0^{+\infty}\Gamma(\lambda,\tau)\gamma(\tau)\pi(\mathrm{d}\tau) + \frac{1}{2}M_1\int_0^{+\infty}\Gamma(\lambda,\tau)\pi(\mathrm{d}\tau) - \bar{X},\ \lambda > 0, \tag{14}$$

$$h = -\frac{1}{w}\int_0^{+\infty}(\gamma(\tau))^2\pi(\mathrm{d}\tau) + \frac{1}{2}M_2\int_0^{+\infty}\Gamma(\theta,\theta)\pi(\mathrm{d}\theta) + M_1\int_0^{+\infty}\gamma(\lambda)\pi(\mathrm{d}\lambda) + \frac{1}{2}\bar{X}^2. \tag{15}$$

In summary, we could reduce an infinite-dimensional equation (9) to the system of finite-dimensional integral equations (13)-(15). This integro-Riccati equation is not found in the literature to the best of our knowledge. The integro-Riccati equation is not solvable analytically, motivating us to employ a numerical method for approximating its solution, which is now the triplet $(\Gamma(\cdot,\cdot),\gamma(\cdot),H)$. Note that the three equations (13)-(15) are effectively decoupled with respect to the three solution variables. With this finding, we can solve them in the order from (13), (14), to (15). This structure also applies to our numerical method.

2.4 Remarks on the Optimality

The optimality of the integro-Riccati equation (13)-(15) follows "formally" by the verification argument [10] based on an Itô's formula for infinite-dimensional SDEs, suggesting that the formula (11) gives an optimal control and $h = H$. To completely prove the optimality, one must deal with the irregularity of the driving noise process that is not of a classical trace class. In particular, possible solutions to the optimality equation (9) should be limited to a functional space such that the non-local term having the Dirac delta is well-defined. The linear-quadratic ansatz (12) meets this requirement, while it is non-trivial whether this holds true in more complicated control problems of supOU processes. One may replace this term by regularizing the correlation of the space-time noise to avoid the well-posedness issue; however, this method

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may not lead to a tractable mathematical model such that the statistical moments and auto-correlation are found explicitly, and hence critically degrades usability of the model in practice. These issues are beyond the scope of this paper because they need sophisticated space-time white noise analysis [e.g., 15].

3. Computation with Actual Data

3.1 Computational Conditions

We show computational examples with an actual discharge data set at an observation station in a perennial mountainous river, Tabusa River, with the mean 2.59 (m^3/s) and variance 61.4 (m^6/s^2). The supOU process was completely identified and statistically examined in Yoshioka [8]. The identified model uses a gamma distribution for π and a tempered stable distribution for v, both of which were determined by a statistical analysis of moments and auto-correlation function. The model correctly fits the auto-correlation with the long-memory behaving as $l^{-0.75}$ for a large time lag l, generates the average, standard deviation, skewness, and kurtosis within the relative error $6.23 \cdot 10^{-3}$ to $8.28 \cdot 10^{-2}$, and furthermore captures the empirical histogram.

The equation (13)-(15) is discretized by the collocation method [10]. The measure π is replaced by the discrete one $\pi(\mathrm{d}\lambda) \to \pi_n(\mathrm{d}\lambda)$ as follows ($1 \le i \le n$):

$$\pi_n(\mathrm{d}\lambda) = \sum_{i=1}^{n+1} c_i \delta_{\{\lambda=\lambda_i\}}, \quad c_i = \int_{\eta_{n,i-1}}^{\eta_{n,i}} \pi(\mathrm{d}\lambda), \quad \lambda_i = \frac{1}{c_i} \int_{\eta_{n,i-1}}^{\eta_{n,i}} \lambda \pi(\mathrm{d}\lambda), \quad \eta_{n,i} = \bar{\eta} \frac{i}{n^\beta} \tag{16}$$

for a fixed resolution $n \in \mathbb{N}$ and parameters $\bar{\eta} > 0$ and $\beta \in (0,1)$, where we define $c_{n+1} = \sum_{i=1}^{n} c_i$ and $\lambda_{n+1} = +\infty$. The parameter $\bar{\eta}$ in the last equation of (16) specifies the degree of domain truncation, while the parameter β modulates the degree of refinement of discretization as the resolution n increases; choosing a larger β means a slower refinement of the discretization.

Replacing π by π_n in (13)-(15) at each node $(\lambda, \theta) = (\lambda_i, \lambda_j)$ ($1 \le i, j \le n$) leads to a system of nonlinear system governing $\Gamma(\lambda_i, \lambda_j)$ ($1 \le i, j \le n$), $\gamma(\lambda_i)$ ($1 \le i \le n$), and $h(=H)$. Instead of directly inverting this system, we add a temporal partial differentials $\frac{\partial \Gamma}{\partial \tau}$ and $\frac{\partial \gamma}{\partial \tau}$ to the left-sided of (13) and (14), respectively with a pseudo-time parameter τ .The temporal discretization is based on a forward Euler scheme with the increment of pseudo-time $1/(24n)$ (day). Stability of numerical solution is maintained with this increment. The system is discretized from initial conditions $\Gamma \equiv 0$ and $\gamma \equiv 0$ until it becomes sufficiently close to a steady state with the increment smaller than 10^{-10} between each successive pseudo-time steps. The quantity H in (15) is then evaluated using the resulting Γ and γ .

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3.2 Computational Results

Because the system (13)-(15) does not admit an explicit solution, we examine convergence of numerical solutions against the following manufactured solution that is obtained by adding appropriate source terms to the right-sides of (13) and (14):

$$\Gamma(\lambda,\theta) = ae^{-b(\lambda+\theta)} \text{ and } \gamma(\lambda) = ce^{-b\lambda}, \ \lambda,\theta > 0 \tag{17}$$

with constants $a,b,c>0$. Namely, we add proper functions $f_1(\lambda,\theta)$ and $f_2(\lambda)$ to (13) and (14) so that the manufactured solution (17) solves the modified equations:

$$0 = -(\theta+\lambda)\Gamma(\theta,\lambda) - \frac{1}{w}\int_0^{+\infty}\Gamma(\theta,\omega)\Gamma(\omega,\lambda)\pi(\mathrm{d}\omega) + 1 + f_1(\lambda,\theta), \ \lambda,\theta > 0 \tag{18}$$

and

$$\begin{aligned} 0 = &-\lambda\gamma(\lambda) - \frac{1}{w}\int_0^{+\infty}\Gamma(\lambda,\tau)\gamma(\tau)\pi(\mathrm{d}\tau) \\ &+\frac{1}{2}M_1\int_0^{+\infty}\Gamma(\lambda,\tau)\pi(\mathrm{d}\tau) - \bar{X} + f_2(\lambda) \end{aligned}, \ \lambda > 0. \tag{19}$$

Here, we still use (15). By the manufactured solution (17), each integral in (18), (19), (15), and hence H is evaluated analytically owing to using the gamma-type π and the tempered stable-type v. The system consisting of the equations (18), (19), (15) is different from the original integro-Riccati equation. However, they share the common integral terms, suggesting that computational performance of the proposed numerical scheme can be examined against the manufactured solution (17).

We set $a = 1.0$, $b = 0.2$, $c = 0.5$, $w = 1$, and $\bar{X} = 15$ (m^3/s), leading to $H = 115.6514$ (m^6/s^2). For the discretization, we fix $\bar{\eta} = 0.05$ (1/h). **Tables 1-2** show the computed H with its relative error (RE) and convergence rate (CR) for $\beta = 0.25$ and $\beta = 0.50$, respectively. The CRs have been computed by the common arithmetic [16]. Similarly, **Tables 3-4** show the computed Γ and γ with its maximum nodal errors (NEs) and CRs for $\beta = 0.25$ and $\beta = 0.50$, respectively.

The numerical solutions converge to the manufactured solutions, verifying the proposed numerical scheme computationally. The CRs of H,Γ,γ are larger than 2.2 for $\beta = 0.25$ and is larger than 0.7 for $\beta = 0.50$ except for the finest level at which the discretization error of c_i,λ_i dominates. The obtained results suggest that choosing the smaller $\beta = 0.25$ is more efficient in this case possibly because the smaller β better harmonizes the domain truncation and node intervals. Note that numerical solutions did not converge to the manufactured solution if $\beta = 0.75$, suggesting an important remark that using a too large β should be avoided.

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Table 1. Computed H with its RE and CR ($\beta = 0.25$).

n	Computed H	RE	CR
10	115.1436	4.39.E-03	2.29.E+00
20	115.5474	3.15.E-02	4.04.E+00
40	115.6451	5.48.E-05	6.88.E+00
80	115.6514	4.64.E-07	3.33.E+00
160	115.6514	4.61.E-08	

Table 2. Computed H with its RE and CR ($\beta = 0.50$).

n	Computed H	RE	CR
10	114.3099	1.16.E-02	7.43.E-01
20	114.8497	6.93.E-03	1.08.E+00
40	115.2710	3.29.E-03	1.57.E+00
80	115.5230	1.11.E-03	2.29.E+00
160	115.6251	2.27.E-04	

Table 3. Computed Γ, γ with their NEs and CRs ($\beta = 0.25$).

n	NE of Γ	NE of γ	CR of Γ	CR of γ
10	1.47.E-01	4.35.E-02	2.22.E+00	2.51.E+00
20	3.15.E-02	7.66.E-03	3.94.E+00	4.58.E+00
40	2.06.E-03	3.21.E-04	6.79.E+00	6.98.E+00
80	1.86.E-05	2.54.E-06	3.21.E+00	-3.51.E-02
160	2.01.E-06	2.60.E-06		

Table 4. Computed Γ, γ with their NEs and CRs ($\beta = 0.50$).

n	NE of Γ	NE of γ	CR of Γ	CR of γ
10	4.89.E-01	1.53.E-01	6.70.E-01	7.29.E-01
20	3.07.E-01	9.20.E-02	8.55.E-01	1.04.E+00
40	1.70.E-01	4.47.E-02	1.24.E+00	1.77.E+00
80	7.21.E-02	1.32.E-02	1.93.E+00	3.08.E+00
160	1.90.E-02	1.56.E-03		

With $n = 100$, $\overline{\eta} = 0.05$ (1/h), and $\beta = 0.25$, we present numerical solutions to the integro-Riccati equation. **Figs. 1-2** show the cases with a small controlling cost $w = 0.5$ and a large cost $w = 5$, respectively. The functional shapes of Γ, γ are common in the two cases, while their magnitudes are significantly different. In both cases, numerical solutions are successfully computed without spurious oscillations.

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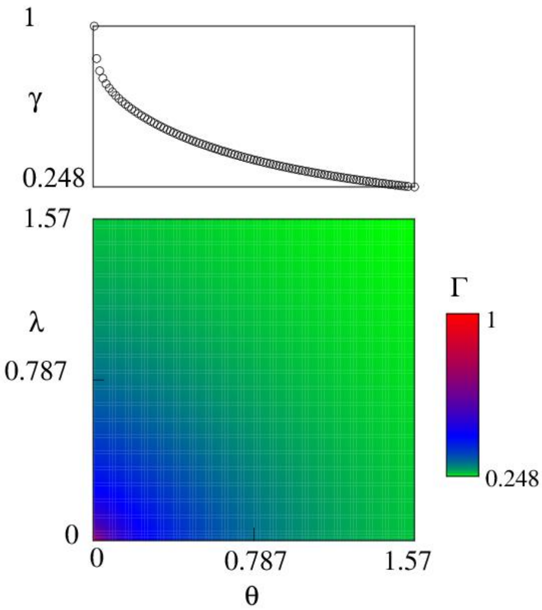

Fig. 1. Computed Γ, γ with $\bar{X} = 10$ (m^3/s) and $w = 0.5$.

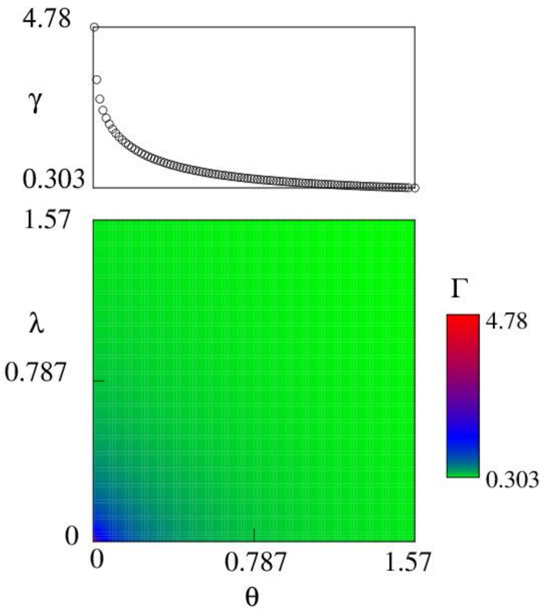

Fig. 2. Computed Γ, γ with $\bar{X} = 10$ (m^3/s) and $w = 5$.

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Finally, **Fig. 3** shows the computed optimized objective H for a variety of couple $(\bar{X},w)$. **Fig. 3** suggests that the optimized objective H is increasing with respect to w. This observation is theoretically consistent with our formulation because increasing the controlling cost should associate a larger value of the objective. Dependence of H on the target discharge $\bar{X}$ is less significant, but seems to be moderately increasing with respect to $\bar{X}$ for each w. This is considered due to that maintaining a higher level is more costly in general for the computed cases here because the minimum discharge is only 0.1 (m^3/s).

In all the computational cases, the computed Γ are positive semi-definite, suggesting that the optimal controls are stabilizable owing to the formula (11). As demonstrated in this paper, the proposed numerical method suffices for computing the optimal control of the infinite-dimensional SDE under diverse conditions.

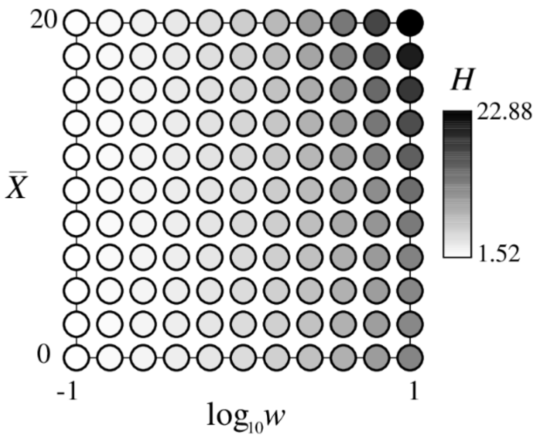

Fig. 3. Computed H for a variety of the couple $(\bar{X},w)$.

4. Conclusion

We presented a novel control problem of an infinite-dimensional SDE arising in environmental management and discussed that it is a highly non-trivial problem due to the noise irregularity. We derived the integro-Riccati equation as a computable optimality equation. Our numerical scheme sufficed to handle this equation.

Currently, we are dealing with a time-periodic control problem of a supOU process as an optimization problem under uncertainty in long-run. Rather difficult is a mathematical justification of the optimality equation under the non-standard space-time noise. Accumulating knowledge from both theoretical and engineering sides would be necessary for correctly understanding control of infinite-dimensional SDEs. Exploring the theoretically optimal discretization of the proposed scheme is also interesting. Applying the proposed framework to weather derivatives [17] based on river hydrological processes will be another future research direction.

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In this paper, we considered a control problem under full information which is a common assumption in most of the stochastic control problems. This means that the observer of the target system has a complete information to construct an optimal control, which is not always technically possible in applications. A possible way to abandon the full-information assumption would be the use of a simpler open-loop control in which the coefficients of optimal controls may be optimized by a gradient descent. Interestingly, for deterministic LQ problems, it has been pointed out that the open-loop problem is often computationally harder than the closed-loop ones [18, 19] because of the non-convex domain of optimization. This finding would apply to stochastic LQ control problems as well. In future, we will compare performance of diverse types of controls including the presented one and the open-loop ones using actual data of environmental management. In particular, modeling and control of coupled hydrological and biochemical dynamics in river environments are of great interest as there exist a huge number of unresolved issues where the proposed stochastic control approach potentially serves as a powerful analysis tool.

We focused on the use of a dynamic programming principle, while the maximum principle can also give an equivalent control formulation based on forward-backward stochastic differential equations. These two principles characterize the same control problem from different viewpoints with each other, naturally leading to different numerical methods for its resolution. Currently, we are investigating an approach from the maximum principle for exploring a more efficient numerical method to compute the LQ and related control problems under uncertainty.

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