1D Painless Multi-Level Automatic Goal-Oriented h and p Adaptive Strategies using a Pseudo-Dual Operator

Felipe V. Caro[1,2][0000−0002−7767−5888], Vincent Darrigrand[3][0000−0001−7818−8216], Julen Alvarez-Aramberri[2][0000−0002−6556−8620], Elisabete Alberdi Celaya[2][0000−0001−9595−0399], and David Pardo[2,1,4][0000−0002−1101−2248]

[1] Basque Center for Applied Mathematics (BCAM), Bilbao, Spain
[2] University of the Basque Country (UPV-EHU), Leioa, Spain
[3] CNRS-IRIT, Toulouse, France
[4] IKERBASQUE, Basque Foundation for Science, Bilbao, Spain

Abstract. The main idea of our Goal-Oriented Adaptive (GOA) strategy is based on performing global and uniform h- or p-refinements (for h- and p-adaptivity, respectively) followed by a coarsening step, where some basis functions are removed according to their estimated importance. Many Goal-Oriented Adaptive strategies represent the error in a Quantity of Interest (QoI) in terms of the bilinear form and the solution of the direct and adjoint problems. However, this is unfeasible when solving indefinite or non-symmetric problems since symmetric and positive definite forms are needed to define the inner product that guides the refinements. In this work, we provide a Goal-Oriented Adaptive (h- or p-) strategy whose error in the QoI is represented in another bilinear symmetric positive definite form than the one given by the adjoint problem. For that purpose, our Finite Element implementation employs a multi-level hierarchical data structure that imposes Dirichlet homogeneous nodes to avoid the so-called hanging nodes. We illustrate the convergence of the proposed approach for 1D Helmholtz and convection-dominated problems.

Keywords: Goal-Oriented Adaptivity · Pseudo-dual Operator · Unrefinements · Finite Element Method · Multi-Level

1 Introduction

One of the main challenges of Finite Element Methods (FEM) is to obtain accurate solutions with low memory requirements. Realistic models are often geometrically complex, and they usually exhibit inhomogeneities. Energy-norm-based adaptive techniques are often employed to model these complex problems. However, many engineering applications demand accurate solutions only in specific domain areas, for example, when the objective is to simulate some measurements

ICCS Camera Ready Version 2022
To cite this paper please use the final published version:
DOI: 10.1007/978-3-031-08754-7_43

at particular receivers. In these scenarios, GOA strategies have shown success for more than twenty years [2,17].

The objective of goal-oriented adaptivity is to build an optimal finite-element grid that minimizes the size of the problem needed to achieve certain tolerance errors for some practical QoI, which is expressed in terms of a linear functional. It has been widely used across different areas of knowledge, including electromagnetic (EM) applications [1,12,16], Cahn–Hilliard-Navier–Stokes systems [10], visco-elasticity [19], and fluid-structure interactions [11]. Traditional approaches represent the error in the QoI by using the direct and adjoint solutions and the global bilinear form of the problem and dividing it in terms of local and computable quantities that are used to guide local refinements (see e.g. [15]).

Here, we follow a different approach. Based on Darrigrand et al. [4], we define an alternative pseudo-dual operator to represent the residual error of the adjoint problem. This new representation, which exhibits better properties than the original bilinear form (e.g., positive definiteness), has proved successful [5,13] and allows to compute the error in the QoI in a way similar to the classical approaches.

The present work combines the energy-based approach introduced in [3] (which uses the data structure proposed by Zander et al. [20,21,22]) and an alternative pseudo-dual operator for representation of the error in the QoI [4]. By doing so, we extend the Darrigrand et al. approach [3] to the context of h- and p-GOA algorithms.

This document is organized as follows: Section 2 describes the GOA strategy and the employed error estimators. Section 3 is devoted to the numerical results, and Section 4 summarizes the main conclusions.

2 Proposed Goal-Oriented Adaptive Algorithms

The h- and p-adaptive algorithms proposed in this work follow the next refinement pattern: first, we perform a global and uniform h- or p-refinement (for the h- and p-adaptive versions, respectively). Then, we perform a coarsening step, where some basis functions are removed. This procedure is illustrated in Algorithm 1, and it was already introduced in [3] in the context of energy-norm adaptivity. The critical part is the coarsening step that we describe next.

Algorithm 1: Adaptive process

```
Input: An initial mesh
Output: A final adapted mesh
while error > tolerance do
    Perform a global and uniform (h or p) refinement;
    Update the error;
    Execute the coarsening step (Algorithm 2) to the mesh;
end
```

ICCS Camera Ready Version 2022
To cite this paper please use the final published version:
DOI: 10.1007/978-3-031-08754-7_43

Optimal unrefinements are performed following an element-by-element approach. Using the multi-level data structures proposed in Zander et al. [20,21,22], we compute element-wise error indicators of all *active* elements, i.e., those elements that do not have sons, or if they do, all new nodes of the sons have homogeneous Dirichlet boundary conditions. These element-based error indicators are one number per element for the h-adaptive version and d numbers for the p-adaptive version, where d is the dimension. The coarsening step procedure is depicted in Algorithm 2. The critical step here is the computation the element-wise error indicators, which we describe in the following subsection.

Algorithm 2: Coarsening processs

Input: A given mesh M.
Output: An unrefined adapted mesh, also denoted as M.
do
  Solve the problem on M;
  Compute the element-wise error indicators for the *active* elements;
  Mark the elements whose indicators are relatively small;
  Update M by unrefining the marked elements;
  If nothing has been marked, escape;
end ;

2.1 Error indicators

We first introduce our boundary value problem in variational form using an abstract formulation:

Find $u \in V$ such that

$$b(u, v) = f(v), \quad \forall v \in V. \tag{1}$$

Here, f is a linear form, b represents a bilinear form and the space V is assumed to be $V = V(\Omega) := \{u \in H^1(\Omega) : u = 0 \text{ on } \Gamma_D,\ u' = 0 \text{ on } \Gamma_N\}$, where Ω is a one-dimensional (1D) computational domain, and Γ_D and Γ_N are the parts of the boundary where we impose homogeneous *Dirichlet* and *Neumann* boundary conditions.

The above *forward* problem has an associated *adjoint* (or *dual*) operator, whose formulation is given by:

Find $w \in V$ such that

$$b(v, w) = l(v), \quad \forall v \in V. \tag{2}$$

ICCS Camera Ready Version 2022
To cite this paper please use the final published version:
DOI: 10.1007/978-3-031-08754-7_43

where l is a linear functional that represents the QoI.

The adjoint problem is often employed in the literature to guide goal-oriented refinements (see, e.g., [14]). However, for the case of indefinite or non-symmetric problems, we further need to introduce an inner product (symmetric and positive definite form) to guide the refinements.

To overcome this issue, we first define $\widetilde{w}$ as a projection of the dual solution w into a given subset of basis functions by simply removing the remaining basis functions' degrees of freedom (DoFs). For the p-adaptive case, the subset of basis functions results from a global p-unrefinement from the current mesh. Thus, $\widetilde{w}$ consists of taking w and replacing the DoF of the highest-order basis functions of each element with zero. In the h-adaptive case, the subset of basis functions results from a global h-unrefinement of the given mesh. Such projections can be trivially implemented in the context of the multi-level data structures proposed in Zander et al. [20,21,22]; but not when using traditional data structures like those described in [7,8,9]. Then, we introduce a *pseudo-dual* bilinear form $\hat{b}$, in our case, defined by the 1D Laplace operator (although it is possible to select other symmetric positive definite bilinear forms). Finally, we solve the following residual-based *pseudo-dual* problem:

Find ε such that

$$\hat{b}(v,\varepsilon) = l(v) - b(v,\widetilde{w}), \quad \forall v \in V. \tag{3}$$

The idea of using an elliptic error representation was already introduced by Romkes et al. [18] and applied by Darrigrand et al. [4] in the context of traditional data structures. However, it required dealing with two grids (fine and coarse) and projection based interpolation operators [6,7,8], which highly complicated its implementation and mathematical analysis. In here, we define problem (3) using simply $\widetilde{w}$ as the projection of w.

Thus, we define E_K as the error indicator associated with element K:

$$E_K \coloneqq \left|\hat{b}(\bar{e}_K,\varepsilon)\right|, \tag{4}$$

where $\bar{e}_K$ is defined for the h-adaptive version as the DoF of u multiplied by the basis function whose support is within the father of the active element K. For the p-version, $\bar{e}_K$ is the DoF of u multiplied by the highest-order basis function whose support is contained within the active element K.

If we assume quasi-orthogonality of our multi-level basis functions, that is, $b(\bar{e}_K,\widetilde{w}) \simeq 0$, then:

$$E_K \coloneqq \left|\hat{b}(\bar{e}_K,\varepsilon)\right| \simeq \left|b(\bar{e}_K,\widetilde{w}) + \hat{b}(\bar{e}_K,\varepsilon)\right| \simeq \left|b(e_K,\widetilde{w}) + \hat{b}(e_K,\varepsilon)\right| = \left|l(e_K)\right|. \tag{5}$$

In the above, we have used equation (3) and defined e_K as the error due to unrefining (in h or p) the element K.

The above error indicators can be extended to 2D and 3D. To account for the possibility of having multiple basis functions in the definition of $\bar{e}_K$, we divide

ICCS Camera Ready Version 2022
To cite this paper please use the final published version:
DOI: 10.1007/978-3-031-08754-7_43

the error indicators by the number of DoFs in $\bar{e}_K$. In 1D, this number is simply one.

3 Numerical Results

To illustrate the performance of our adaptive strategies, we consider two problems, governed by Helmholtz and convection-dominated equations. We provide the evolution of the relative error in the QoI for h- and p-adaptivity and for different values of the PDE parameters. To define the relative error in the QoI, we compute $l(u)$ on a globally refined mesh. Then, we describe the relative error as follows:

$$e_{\text{rel}} = \frac{|l(u) - l(u_c)|}{|l(u)|} \cdot 100, \tag{6}$$

where $l(u_c)$ is the QoI associated with the adapted mesh.

3.1 Helmholtz goal-oriented problem

Let us consider the following wave propagation problem:

Find u such that,

$$-u'' - k^2 u = \mathbb{1}_{(0,\frac{2}{5})} \text{ in } (0,1), \tag{7}$$

$$u(0) = 0, \tag{8}$$

$$u'(1) = 0. \tag{9}$$

We define the QoI as $l(u) = 5 \cdot \int_{\frac{3}{5}}^{\frac{4}{5}} u \, dx$. Figures 1 and 2 show the evolution of the relative error in the QoI by using h- and p-adaptivity, respectively. Note that the larger the number of DoFs per wavelength, the faster the decrease of the relative error in the QoI. For example, in Figure 1, for $k = 7 \cdot 2\pi$, 10 DoFs per wavelength are sufficient to enter into the so called asymptotic regime. In contrast, for $k = 28 \cdot 2\pi$, we need to consider at least 40 DoFs per wavelength. In Figure 2, we select the initial mesh size such that the number of DoFs per wavelength is at least 3. This way, we satisfy the Nyquist rate. Both Figures 1 and 2 show optimal convergence rates in both h- and p-adaptivity. As a curiosity, we observe that the curves in Figure 1 are parallel, while the ones in Figure 2 coincide. This occurs due to dispersion (pollution) error, which quickly disappears with the p-method.

ICCS Camera Ready Version 2022
To cite this paper please use the final published version:
DOI: 10.1007/978-3-031-08754-7_43

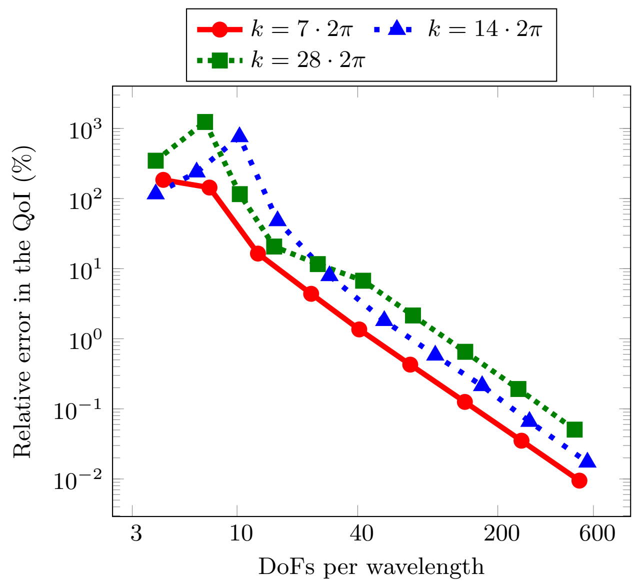

Fig. 1: Evolution of the relative error in the QoI by using h-adaptivity. Initial mesh size $h = \frac{1}{30}$ and uniform $p = 1$.

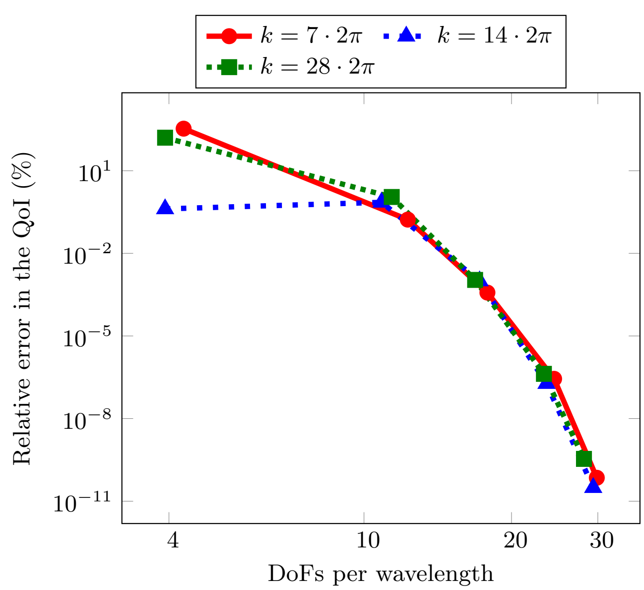

Fig. 2: Evolution of the relative error in the QoI by using p-adaptivity. Uniform mesh size $h = \frac{1}{30}$.

ICCS Camera Ready Version 2022
To cite this paper please use the final published version:
DOI: 10.1007/978-3-031-08754-7_43

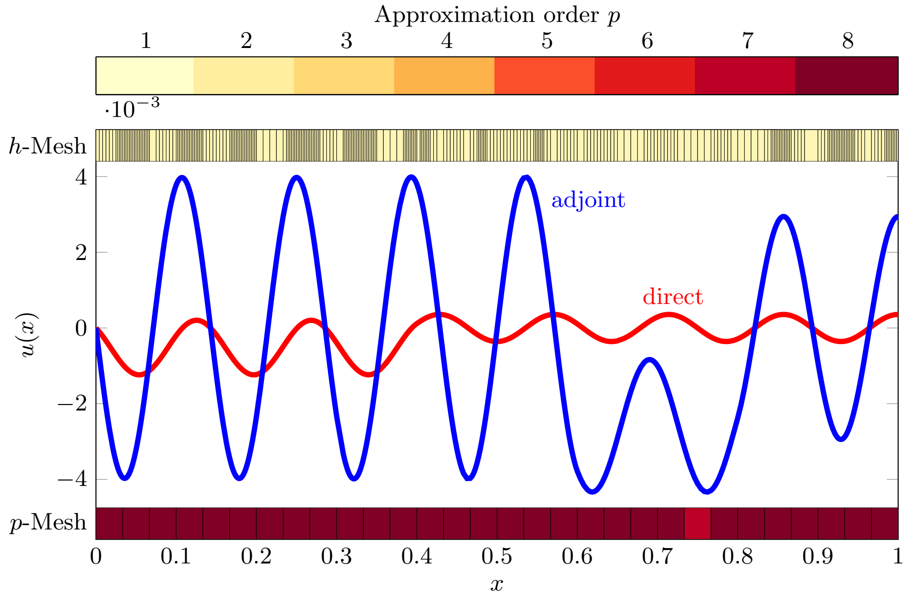

Fig. 3: Solutions with $k = 7 \cdot 2\pi$ problem as given after the h-adaptive process.

Figure 3 shows the solutions for the case $k = 7 \cdot 2\pi$. We also provide the corresponding h- and p-adaptive meshes. For the p-adaptive mesh, we show the mesh obtained in the 6th iteration, containing high approximation orders. To visualize the h-adaptive mesh, we show the mesh obtained in the 5th iteration. Finally, we show the solutions corresponding to the 6th iteration, which contains small localized values of the mesh size h.

3.2 Convection-dominated goal-oriented problem

Let us consider the boundary value problem associated with steady convective-diffusive transport:

Find u such that,

$$\begin{aligned} -\varepsilon u'' + \sigma \cdot u' &= \mathbb{1}_{(0,1)} \text{ in } (0,1), \\ u(0) = u(1) &= 0, \end{aligned} \tag{10}$$

with $\sigma = 1$, and $0 < \varepsilon \ll 1$ the diffusive coefficient. We define the QoI as $l(u) = 5 \cdot \int_{\frac{4}{5}}^{1} \nabla u \, dx$.

In Figures 4 and 5, we represent the evolution of the relative error in the QoI by using h- and p-adaptivity, respectively. We observe optimal convergence rates. We note that the smaller the diffusive coefficient ε, the larger the number of DoFs to reach the convergence rates.

ICCS Camera Ready Version 2022
To cite this paper please use the final published version:
DOI: 10.1007/978-3-031-08754-7_43

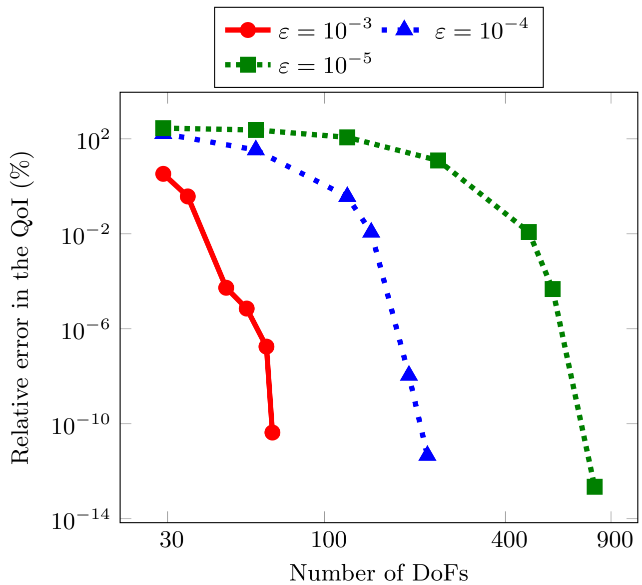

Fig. 4: Evolution of the relative error in the QoI by using h-adaptivity. Initial mesh size $h = \frac{1}{30}$ and uniform $p = 1$.

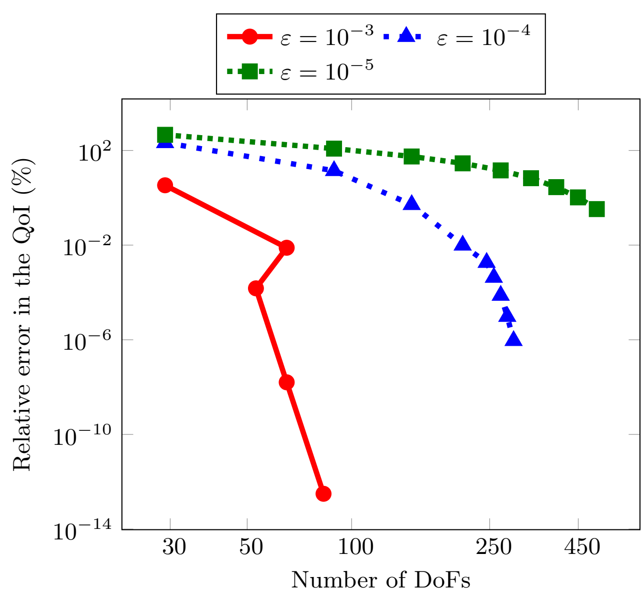

Fig. 5: Evolution of the relative error in the QoI by using p-adaptivity. Uniform mesh size $h = \frac{1}{30}$.

ICCS Camera Ready Version 2022
To cite this paper please use the final published version:
DOI: 10.1007/978-3-031-08754-7_43

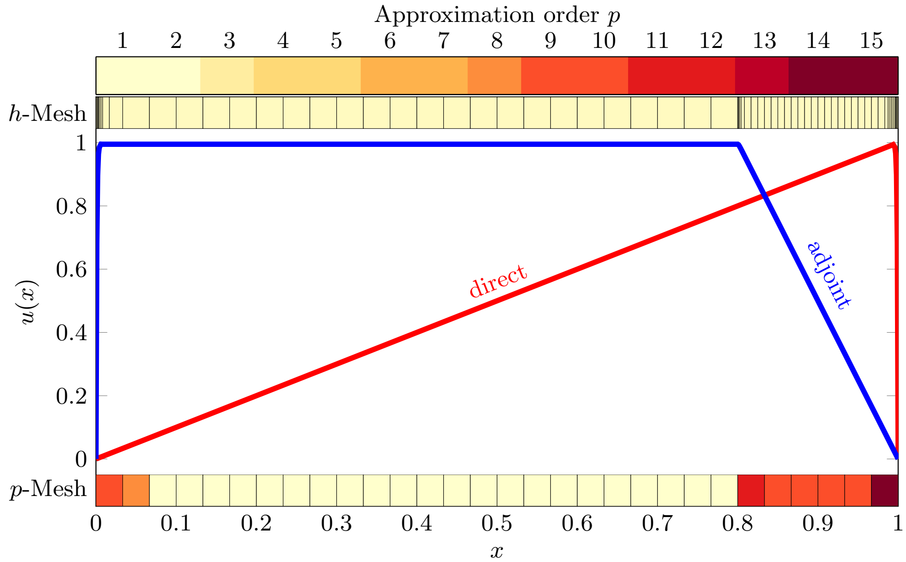

Fig. 6: Solutions with $\varepsilon = 10^{-3}$ problem as given after the h-adaptive process.

Figure 6 shows the solutions for the case $\varepsilon = 10^{-3}$. We also provide the meshes and the solutions corresponding to the last iteration in both h and p cases.

4 Conclusions

We propose h- and p-GOA strategies for possibly non-elliptic problems. These adaptive algorithms are simple-to-implement because they take advantage of multi-level data structures with hierarchical basis functions that avoid the problem of hanging nodes altogether.

The main idea of this approach consists of performing first a global and uniform refinement followed by a coarsening step, where some basis functions are removed. To select which basis functions are eliminated, we employ a representation of the error in the QoI that uses an unconventional symmetric and positive definite bilinear form.

1D numerical results show a proper convergence for Helmholtz and convection-dominated GOA problems when using the Laplace operator's pseudo-dual problem. This adaptive strategy can be easily extended to 2D and 3D problems, and it can be further exploited in other indefinite and/or non-symmetric problems.

References

1. Alvarez-Aramberri, J., Pardo, D., Barucq, H.: A secondary field based hp-finite element method for the simulation of magnetotelluric measurements. Journal of

ICCS Camera Ready Version 2022
To cite this paper please use the final published version:
DOI: 10.1007/978-3-031-08754-7_43

Computational Science **11**, 137–144 (2015)
2. Becker, R., Rannacher, R.: An optimal control approach to a posteriori error estimation in finite element methods. Acta numerica **10**, 1–102 (2001)
3. Darrigrand, V., Pardo, D., Chaumont-Frelet, T., Gómez-Revuelto, I., Garcia-Castillo, L.E.: A painless automatic hp-adaptive strategy for elliptic problems. Finite Elements in Analysis and Design **178**, 103424 (2020). https://doi.org/https://doi.org/10.1016/j.finel.2020.103424
4. Darrigrand, V., Pardo, D., Muga, I.: Goal-oriented adaptivity using unconventional error representations for the 1D Helmholtz equation. Computers & Mathematics with Applications **69**(9), 964 – 979 (2015). https://doi.org/http://dx.doi.org/10.1016/j.camwa.2015.03.006, http://www.sciencedirect.com/science/article/pii/S0898122115001017
5. Darrigrand, V., Rodríguez-Rozas, Á., Muga, I., Pardo, D., Romkes, A., Prudhomme, S.: Goal-oriented adaptivity using unconventional error representations for the multi-dimensional Helmholtz equation. International Journal for Numerical Methods in Engineering **113**(1), 22–42 (2018). https://doi.org/10.1002/nme.5601, http://dx.doi.org/10.1002/nme.5601, nme.5601
6. Demkowicz, L., Rachowicz, W., Devloo, P.: A fully automatic *hp*-adaptivity. In: Proceedings of the Fifth International Conference on Spectral and High Order Methods (ICOSAHOM-01) (Uppsala). vol. 17, pp. 117–142 (2002). https://doi.org/10.1023/A:1015192312705, http://dx.doi.org/10.1023/A:1015192312705
7. Demkowicz, L.: Computing with *hp*-adaptive finite elements. Vol. 1. One and two dimensional elliptic and Maxwell problems. Applied Mathematics and Nonlinear Science Series, Chapman & Hall/CRC, Boca Raton, FL (2007). https://doi.org/10.1201/9781420011692, http://dx.doi.org/10.1201/9781420011692
8. Demkowicz, L., Kurtz, J., Pardo, D., Paszyński, M., Rachowicz, W., Zdunek, A.: Computing with *hp*-adaptive finite elements. Vol. 2. Frontiers: three dimensional elliptic and Maxwell problems with applications. Applied Mathematics and Nonlinear Science Series, Chapman & Hall/CRC, Boca Raton, FL (2008)
9. Demkowicz, L., Oden, J.T., Rachowicz, W., Hardy, O.: Toward a universal hp adaptive finite element strategy, part 1. constrained approximation and data structure. Computer Methods in Applied Mechanics and Engineering **77**(1-2), 79–112 (1989)
10. Hintermüller, M., Hinze, M., Kahle, C., Keil, T.: A goal-oriented dual-weighted adaptive finite element approach for the optimal control of a nonsmooth cahn–hilliard–navier–stokes system. Optimization and Engineering **19**(3), 629–662 (2018)
11. Jhurani, C., Demkowicz, L.: Multiscale modeling using goal-oriented adaptivity and numerical homogenization. part i: Mathematical formulation and numerical results. Computer Methods in Applied Mechanics and Engineering **213**, 399–417 (2012)
12. Key, K.: Mare2dem: a 2-d inversion code for controlled-source electromagnetic and magnetotelluric data. Geophysical Journal International **207**(1), 571–588 (2016)
13. Muñoz-Matute, J., Alberdi, E., Pardo, D., Calo, V.M.: Time-domain goal-oriented adaptivity using pseudo-dual error representations. Computer Methods in Applied Mechanics and Engineering **325**, 395–415 (2017)
14. Oden, J.T., Prudhomme, S.: Goal-oriented error estimation and adaptivity for the finite element method. Comput. Math. Appl. **41**(5-6), 735–756 (2001). https://doi.org/10.1016/S0898-1221(00)00317-5, http://dx.doi.org/10.1016/S0898-1221(00)00317-5

15. Ovall, J.S.: Asymptotically exact functional error estimators based on superconvergent gradient recovery. Numerische Mathematik **102**(3), 543–558 (2006)
16. Pardo, D., Demkowicz, L., Torres-Verdín, C., Paszynski, M.: Two-dimensional high-accuracy simulation of resistivity logging-while-drilling (LWD) measurements using a self-adaptive goal-oriented *hp* finite element method. SIAM J. Appl. Math. **66**(6), 2085–2106 (2006). https://doi.org/10.1137/050631732, http://dx.doi.org/10.1137/050631732
17. Prudhomme, S., Oden, J.T.: On goal-oriented error estimation for elliptic problems: application to the control of pointwise errors. Comput. Methods Appl. Mech. Engrg. **176**(1-4), 313–331 (1999). https://doi.org/10.1016/S0045-7825(98)00343-0, http://dx.doi.org/10.1016/S0045-7825(98)00343-0
18. Romkes, A., Oden, J.T.: Adaptive modeling of wave propagation in heterogeneous elastic solids. Computer methods in applied mechanics and engineering **193**(6-8), 539–559 (2004)
19. Van Der Zee, K.G., Tinsley Oden, J., Prudhomme, S., Hawkins-Daarud, A.: Goal-oriented error estimation for cahn–hilliard models of binary phase transition. Numerical Methods for Partial Differential Equations **27**(1), 160–196 (2011)
20. Zander, N., Bog, T., Elhaddad, M., Frischmann, F., Kollmannsberger, S., Rank, E.: The multi-level *hp*-method for three-dimensional problems: Dynamically changing high-order mesh refinement with arbitrary hanging nodes. Computer Methods in Applied Mechanics and Engineering **310**, 252 – 277 (2016). https://doi.org/https://doi.org/10.1016/j.cma.2016.07.007, http://www.sciencedirect.com/science/article/pii/S0045782516307289
21. Zander, N., Bog, T., Kollmannsberger, S., Schillinger, D., Rank, E.: Multi-level *hp*-adaptivity: high-order mesh adaptivity without the difficulties of constraining hanging nodes. Computational Mechanics **55**(3), 499–517 (Mar 2015). https://doi.org/10.1007/s00466-014-1118-x, https://doi.org/10.1007/s00466-014-1118-x
22. Zander, N.D.: Multi-level *hp*-FEM: dynamically changing high-order mesh refinement with arbitrary hanging nodes. Ph.D. thesis, Technische Universität München (2017)

ICCS Camera Ready Version 2022
To cite this paper please use the final published version:
DOI: 10.1007/978-3-031-08754-7_43