# Pseudo-Newton Method with Fractional Order Derivatives

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**Abstract.** Recently, the pseudo-Newton method was proposed to solve the problem of finding the points for which the maximal modulus of a given polynomial over the unit disk is attained. In this paper, we propose a modification of this method, which relies on the use of fractional order derivatives. The proposed modification is evaluated twofold: visually via polynomiographs coloured according to the number of iterations, and numerically by using the convergence area index, the average number of iterations and generation time of polynomiographs. The experimental results show that the fractional pseudo-Newton method for some fractional orders behaves better in comparison to the standard algorithm.

**Keywords:** pseudo-Newton method · Riemann–Liouville derivative · Caputo derivative · dynamics.

## 1 Introduction

Newton's method is one of the most famous and important algorithms in numerical analysis. It has a local quadratic convergence and is undefined for critical points. This simple algorithm has a long history and ample bibliography [13]. In recent years many modifications of Newton's method have been proposed.

An interesting modification of the Newton's method is the pseudo-Newton method [8]. That method effectively finds the local maximal values of the modulus of complex polynomials over the unit disc on the complex plane.

In recent years, various fractional derivatives have become an intensive field of study in root-finding area. The first method, in which the classical derivative was replaced by the fractional ones, was the Newton's method [5]. Then, the Newton-type method with convergence of order $\nu$ was proposed [1], where $\nu$ is the order of the fractional derivative. Next, the use of fractional derivatives and various iteration processes in the Newton's method was shown [6]. Then, a variant of Chebyshev's method [2] and a two-step iterative scheme with fractional derivatives [3] were introduced.

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In this paper, we replace the classic derivative with the fractional derivatives in the pseudo-Newton method. This leads to a new class of pseudo-Newton's methods – fractional pseudo-Newton methods. The performed numerical experiments suggest that for some values of $\nu$ the fractional pseudo-Newton method is better in comparison to the standard pseudo-Newton one.

The paper is organised as follows. In Sec. 2, the definitions of the Riemann–Liouville and Caputo derivatives, are presented. In Sec. 3, the pseudo-Newton method, introduced in [8], is described. In Sec. 4, the application of the fractional derivatives into the pseudo-Newton method is proposed. In Sec. 5, the experimental results are shown. Finally, Sec. 6 concludes this paper.

## 2 Fractional Derivatives

Integer order derivatives and integrals are commonly known and used. As a natural generalisation, the fractional derivative was introduced [10]. It can be defined in many ways. In this paper, we use Riemann–Liouville and Caputo derivatives as the most commonly used ones. We recall their definitions [10,11].

Let $\Gamma$ be the well-known gamma function. The Riemann–Liouville derivative (RL-derivative) of order $\nu \in (n-1, n]$, $n \in \mathbb{N}$ is defined as:

$$D_{RL}^{\nu} f(t) := \begin{cases} \frac{1}{\Gamma(n-\nu)} \frac{d^n}{dt^n} \int_0^t \frac{f(\tau)}{(t-\tau)^{\nu+1-n}} d\tau, & \text{if } \nu \in (n-1, n), \\ \frac{d^n}{dt^n} f(t), & \text{if } \nu = n. \end{cases} \tag{1}$$

We also recall that the Caputo derivative (C-derivative) of order $\nu \in (n-1, n]$, $n \in \mathbb{N}$, of a real–valued function $f$ is defined as:

$$D_{C}^{\nu} f(t) := \begin{cases} \frac{1}{\Gamma(n-\nu)} \int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\nu+1-n}} d\tau, & \text{if } \nu \in (n-1, n), \\ \frac{d^n}{dt^n} f(t), & \text{if } \nu = n. \end{cases} \tag{2}$$

Both of these fractional derivatives are linear.

In this paper, we calculate the fractional derivatives of polynomials. Thus, to determine them, we can only consider monomial $t^m$, thanks to linearity. So,

$$D_{RL}^{\nu} t^m = \frac{\Gamma(m+1)}{\Gamma(m-\nu+1)} t^{m-\nu}, \ D_{C}^{\nu} t^m = \begin{cases} \frac{\Gamma(m+1)}{\Gamma(m-\nu+1)} t^{m-\nu}, & \text{if } m > n-1, \\ 0, & \text{if } m \leq n-1, \end{cases} \tag{3}$$

where $\nu \in (n-1, n)$, $n \in \mathbb{N}$, $m \in \mathbb{R}$.

Let us note that for a constant function and $\nu \neq 1$ we obtain that $D_{RL}^{\nu} c \neq 0$ and $D_{C}^{\nu} c = 0$. So, these derivatives are not equal.

So far, we presented the fractional derivatives of functions defined on $\mathbb{R}$. But we are going to use them on $\mathbb{C}$. However, we cannot replace the real variable $t$ by a complex variable $z$ in the definitions in (1) and (2) because of the multivaluedness of expressions that are present under integrals in both derivatives. Nevertheless, in the case of analytic functions the formulas for the RL- and C-derivative for monomial $z^m$ are the same as in (3), but only for a complex variable

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$z$ such that $z \in \mathbb{C} \setminus \{c \in \mathbb{C} : Im(c) = 0 \wedge Re(c) < 0\}$, and $m \neq -1, -2, -3, \ldots$. This additional assumption is related to the branch cut line that is needed to eliminate the multi-valuedness of $z^m$ if $z \in \mathbb{C}$ and $m \in \mathbb{R}$.

## 3 The Pseudo-Newton Method

Let $p$ be a non-constant complex polynomial over the unit disk $D = \{z \in \mathbb{C} : |z| \leq 1\}$. Let us then consider the problem of finding the points for which we attain the maximal modulus over $D$, i.e., $\|p\|_\infty = \max\{|p(z)| : z \in D\}$.

According to the Maximum Modulus Principle [7], $\|p\|_\infty$ is attained at the boundary of $D$ [8]. Moreover, we have that a point $z_* \in D$ is a local maximum of $|p(z)|$ if and only if

$$z_* = \left(\frac{p(z_*)}{p'(z_*)}\right) \Big/ \left(\left|\frac{p(z_*)}{p'(z_*)}\right|\right). \tag{4}$$

Equation (4) is the test for checking if $z_*$ is a local maximum of $|p(z)|$ over $D$. But instead of solving (4) one can solve the following equation [8,9]:

$$G(z) = p(z)|p'(z)| - zp'(z)|p(z)| = 0. \tag{5}$$

All the solutions of (5) are the fixed points of (4). To find them one can use the pseudo-Newton method [8]. This method has the following form:

$$z_{n+1} = z_n - \frac{G_n(z_n)}{G_n'(z_n)}, \quad n = 0, 1, 2, \ldots, \tag{6}$$

where $z_0 \in \mathbb{C}$ is a given starting point and $G_n(z) = p(z)|p'(z_n)| - zp'(z)|p(z_n)|$. Let us observe that the functions $G_n$ are easily differentiable with respect to $z$ because the modules in them are constant values.

The proof of convergence of the pseudo-Newton method can be found in [12], where this method is converted to some equivalent convergent Newton-like method. Moreover, the pseudo-Newton method can be easily generalised to higher-order methods [4].

## 4 The Pseudo-Newton Method with Fractional Derivatives

In recent years, many applications of fractional derivatives appeared in the literature. A good example is the use of Riemann–Liouville and Caputo derivatives in the classical Newton's method [1,5]. In this section, we present a similar combination of the pseudo-Newton method with the fractional order derivatives.

Let us denote by $D_*^\nu$ any of the two considered fractional derivatives, i.e., $D_{RL}^\nu$ or $D_C^\nu$. By replacing the classical first derivative $G_n'$ in (6) by $D_*^\nu$, we get

$$z_{n+1} = z_n - \frac{G_n(z_n)}{D_*^\nu G_n(z_n)}, \quad n = 0, 1, 2, \ldots. \tag{7}$$

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Such defined methods are called fractional pseudo-Newton methods.

In these methods, we need to calculate the derivative $D_*^\nu G_n(z)$. When we look closely at the form of function $G_n$, we can notice that for a fixed $n$ the values of $|p'(z_n)|$, $|p(z_n)|$ are constant. Therefore, the terms $p(z)|p'(z_n)|$ and $zp'(z)|p(z_n)|$ are polynomials of argument $z$. So, due to the linearity property of $D_{RL}^\nu$ and $D_C^\nu$, the derivative $D_*^\nu G_n(z)$ has the following form

$$D_*^\nu G_n(z) = |p'(z_n)| D_*^\nu(p(z)) - |p(z_n)| D_*^\nu(zp'(z)). \tag{8}$$

## 5 Numerical Results

In this section, we present the numerical results of application of the proposed methods in practice. We start by presenting the polynomiographs that show the speed of convergence and the dynamics of the proposed method graphically. Then, we show the dependencies between some numerical measures and the order of the considered fractional derivatives.

To generate a polynomiograph in the given area, we take each point of this area as a starting point for (7). Then, we map the number of the performed iterations to a colour by using the colour map from Fig. 1. Basing on the polynomiograph, we compute the following numerical measures: the average number of iterations (ANI) in the considered area, the convergence area index (CAI, i.e., the ratio of the number of the points that converged to the number of all points in the considered area), and the generation time of the polynomiograph.

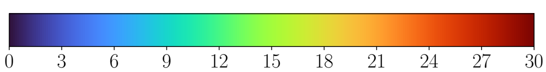


**Fig. 1.** The colour map used in the experiments.

The experiments were performed for a number of polynomials, but due to the lack of space, we present here the complete results (i.e., the polynomiographs and the plots of numerical measures) only for $p_4(z) = z^4 - 10z^2 + 9$. To generate the polynomiographs we used the following parameters: the area is fixed as $[-3, 3]^2$, the maximal number of iterations equals to 30, accuracy $\varepsilon = 0.001$, and image resolution is $800 \times 800$ pixels. The experiments were performed on the computer with: Intel i5-9600K (@ 3.70 GHz) processor, 32 GB DDR4 RAM, and Windows 10 (64-bit). The software was implemented in Processing.

We start with the polynomiographs for $p_4$. In Fig. 2, we see the polynomiograph generated by the pseudo-Newton method with the classical derivative. Next, in Figs. 3 and 4, we see the polynomiographs obtained with the fractional versions of the pseudo-Newton method. We have chosen only the most representative ones. The presented dynamics is related to the number of iterations needed to achieve the maximum modulus of polynomials via the investigated

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algorithms. In general, the blue colour in the polynomiographs means quick convergence, the green one means average convergence and the red colour denotes slow convergence. The polynomiographs from Figs. 3, 4 show less dynamic in comparison to the reference one from Fig. 2. Indeed, one can see that the larger or lower the value of $\nu$ related to $\nu = 1$ the slower the convergence of the polynomiographs (we can observe more red colour). Additionally, careful analysis of them suggests that there exist values of $\nu$ for which the fractional pseudo-Newton methods could be better (in the mean of higher CAI and lower ANI values) in comparison to the pseudo-Newton method with the classic derivative.

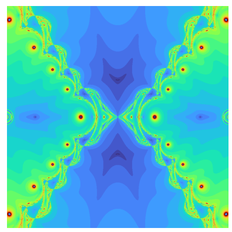
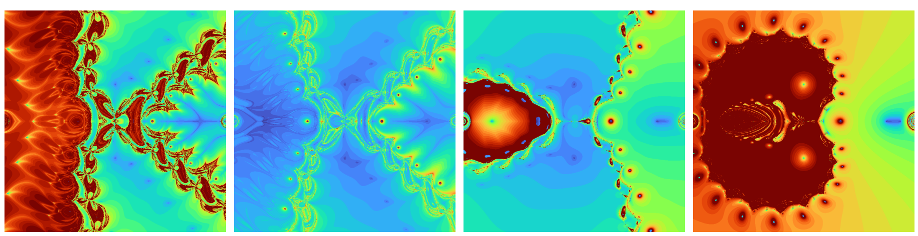


**Fig. 2.** The dynamics for the classical derivative ($\nu = 1$) for $p_4$.



(a) $\nu = 0.775$ (b) $\nu = 0.850$ (c) $\nu = 1.2$ (d) $\nu = 1.625$

**Fig. 3.** Examples of dynamics for the RL-derivative for different values of $\nu$ for $p_4$.

The dependencies between the considered numerical measures (ANI, CAI, and generation time) and the order $\nu$ of the fractional derivatives for $p_4$ are presented in Fig. 5. The best values of the numerical measures are the following:

- classical derivative – ANI: 8.578, CAI: 0.998, time: 1.180 s,
- RL–derivative – min. ANI: 8.197 ($\nu = 0.93$), max. CAI: 0.999 ($\nu = 0.835$), min. time: 5.316 s ($\nu = 0.940$),
- C–derivative – min. ANI: 7.747 ($\nu = 0.850$), max. CAI: 0.999 ($\nu = 0.805$), min. time: 4.387 s ($\nu = 0.890$).

From the results presented above and the plots shown in Fig. 5, one can see that by using fractional derivatives one can decrease the value of ANI and

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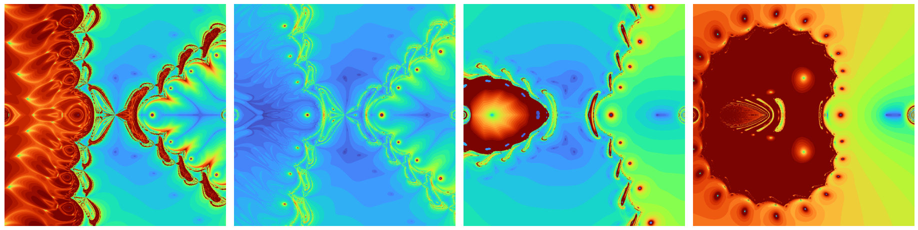
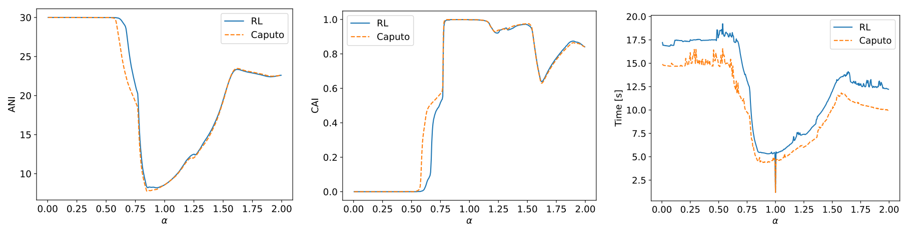


(a) $\nu = 0.775$ (b) $\nu = 0.850$ (c) $\nu = 1.2$ (d) $\nu = 1.625$

**Fig. 4.** Examples of dynamics for the C-derivative for different values of $\nu$ for $p_4$.




(a) (b) (c)

**Fig. 5.** The plots of (a) ANI, (b) CAI, and (c) time (in seconds), for polynomial $p_4$.

improve the convergence (higher values of CAI). The decrease of ANI and the increase of CAI compared to the classical case can be observed for $\nu < 1$, but in the neighbourhood of 1. For $\nu > 1$, the results for the fractional case are worse than for the classical derivative. Unfortunately, the generation time of the polynomiographs via the fractional pseudo-Newton method cannot be improved. In general, calculation cost is higher for fractional derivatives compared to classical derivatives. It is because in the case of the classical derivatives, we raise to a power with only an integer exponent, whereas in the fractional case we raise to real-valued exponents that is more computationally expensive. Additionally, one can observe that for C–derivative, the plots of ANI in some intervals below $\nu = 1$ are lying below those for RL–derivative. The same occurs for time plots. Moreover, for CAI plots it is conversely. This generally denotes that C–derivatives should be preferred over the RL–derivatives since the former ones converge faster.

## 6 Conclusions

In this paper, we proposed the use of the fractional derivatives instead of the classical one in the pseudo-Newton method. The experimental results showed that the proposed approach can improve the standard pseudo-Newton method in some aspects. Namely, for some values of order $\nu$, the value of ANI is lower and the value of CAI is higher. Unfortunately, the generation time of poly-

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nomiographs is higher, which is clear because we must perform more computationally complex calculations for the fractional derivatives.

Similar investigations could be performed for the higher-order pseudo-methods [4]. It could be also interesting to check the behaviour of further modifications of the fractional pseudo-Newton methods obtained by replacing the standard Picard iteration with various types of iterations [4,6].

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