PIES with trimmed surfaces for solving elastoplastic boundary problems

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Abstract. The paper presents the strategy for solving elastoplastic problems using a parametric integral equation system (PIES) and a trimming technique. It allows even complex shapes of a plastic zone to be modeled with a single surface and a set of trimming curves. New schemes for integration and approximation of solutions are developed to include changed requirements. However, both of them have kept their advantages. Some examples are solved, and the obtained results are compared with analytical solutions and those received from other numerical methods.

Keywords: PIES · elastoplastic problems · trimmed surfaces.

1 Introduction

Next to FEM [1] and BEM [2], PIES [3, 4] is used for solving boundary problems e.g. elastoplastic. The main advantage of PIES, in this case, is the elimination of discretization of the predicted yield region (by cells in BEM) or the whole domain (by finite elements in FEM). Instead, the plastic zone is modeled by a surface (e.g. a Bezier surface [5]), which requires a small number of control points to be defined. Modification of the surface is also very easy because is limited to changing the positions of some control points. Moreover, the integrals are calculated globally over the whole domain, and the approximation of plastic strains is performed in the same way. This distinguishes PIES from the approaches in which Bezier surfaces are used for FEM modeling (e.g. [6]), as FEM still requires discretization into elements for numerical integration or solution approximation.

The surfaces, however, have some limitations. If the expected yield region is complex, it is difficult to deal with it using one surface. A solution to this situation is the trimming technique, which allows modeling arbitrary regions using trimming curves. The initial domain is defined by the bilinear surface and the desired shape is created by the set of trimming curves. The proposed approach has one additional advantage, very often there is no need to determine the expected shape of the plastic area, as the entire created domain is treated as potentially plastic. This prevents cases where an incorrectly predicted shape necessitates resolving the problem. The trimming technique is often used in isogeometric FEM [7, 8] since the whole domain is always defined, but as is mentioned above discretization into elements is still present.

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The main aim of the paper is to develop an approach for solving elastoplastic problems with any shape of plastic regions using the trimming technique. The proposed strategy requires modifying the plastic strain approximation method along with determining the necessary interpolation points and adjusting the integration method currently used in PIES. For approximation, the inverse distance weighting (IDW) method [9] is used, performed only on points designated by the projection scheme [7], while for integration a modified transformation technique [2] for calculating singular integrals is applied. Three test examples are included, with the results confirming the effectiveness of the approach.

2 PIES for elastoplastic problems

The PIES formula for solving 2D elastoplastic problems using initial stress formulation can be presented in the following form [3, 4]

$$0.5\dot{\boldsymbol{u}}_l(\bar{s}) = \sum_{j=1}^{n} \int_{s_{j-1}}^{s_j} \left\{ \mathbf{U}^*_{lj}(\bar{s},s)\dot{\boldsymbol{p}}_{\boldsymbol{j}}(s) - \mathbf{P}^*_{lj}(\bar{s},s)\dot{\boldsymbol{u}}_{\boldsymbol{j}}(s) \right\} J_j(s)ds + \int_{\Omega} \boldsymbol{\sigma}^*_l(\bar{s},\boldsymbol{y})\dot{\boldsymbol{\varepsilon}}^p(\boldsymbol{y})d\Omega(\boldsymbol{y}). \quad (1)$$

The fundamental solutions for displacement $\mathbf{U}^*_{lj}(\bar{s},s)$, traction $\mathbf{P}^*_{lj}(\bar{s},s)$ and stress $\boldsymbol{\sigma}^*_l(\bar{s},\boldsymbol{y})$ are presented explicitly in [3, 4]. The functions $\dot{\boldsymbol{u}}_{\boldsymbol{j}}(s)$, $\dot{\boldsymbol{p}}_{\boldsymbol{j}}(s)$ and $\dot{\boldsymbol{\varepsilon}}^p(\boldsymbol{y})$ describe the distribution of displacements, tractions on the boundary and plastic strains in the domain, respectively. Both the boundary and the domain in PIES are defined in a parametric reference system using curves and surfaces. The boundary is composed of n segments represented by any curves Γ, whose beginning and end are determined by s_{j-1} and s_j. The domain is modeled by any surface Ω. Variables s and $\bar{s}$ are parameters in the mentioned parametric reference system, $J_j(s)$ is the Jacobian, $\mathbf{y} \in \Omega$ and $l, j = 1..n$.

3 The plastic zone modeled by the trimmed surface

In elastoplastic problems, it is not enough to define the boundary itself, as it is not able to create a domain representing a plastic area. In FEM [1] to form such a domain, regardless of the problem, the whole body is divided into finite elements (Fig. 1a). In BEM [2] only the region with predicted plastic strains is modeled by cells (Fig. 1b). It is more effective, but still requires discretization of part of the domain and it can be troublesome if the region needs to be remodeled. PIES also requires defining only the yield region, but it is represented as a whole, without discretization, using a Bezier surface. In other words, the Bezier surface is like one global element as shown in Fig. 1c. What is important, the defined initially surface does not change when solving the problem (same as in BEM), and the plastic zone is determined by yielded interpolation points at each increment.

As can be seen in Fig. 1a,b,c, the strategy used in PIES is more effective than those applied in FEM and BEM. Posing a large number of elements requires

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declaring many nodes and modifying them in case of any shape changes. In PIES the whole expected plastic region is defined by the small number of control points of the surface [5]. Its modification is also very simple, it is enough to change the position of individual points. The only problem that arises is the complexity of the plastic zone, especially when it cannot be modeled with a single Bezier surface. The strategy proposed in this paper is to use the trimming technique.

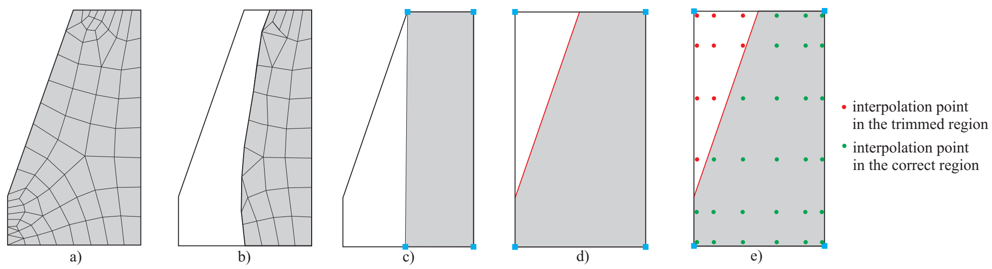

Fig. 1. Modeling in: a) FEM, b) BEM, c) PIES d) PIES with the trimmed surface and e) interpolation points arrangement in the correct and trimmed regions

The main idea of the trimming technique is to use a bilinear Bezier surface, from which the expected domain is determined by trimming curves (Bezier curves in this paper). A certain orientation rule should be followed when defining them. It says that the correct region is to the left of the curve. Fig. 1d presents the characteristic of the trimming technique using the geometry from Fig. 1c. As can be seen, the bilinear Bezier surface is modeled using four control points (blue squares) and the trimming curve (red line) designates the right area (grey region). Bezier curves of any degree can be used for trimming. In addition to the obvious advantage of being able to model complex domains, this strategy in many cases relieves the need to predict the shape of the plastic region, as the whole body area can be treated as potentially plastic.

4 Numerical solving of PIES with the trimmed surface

To solve PIES (1), functions $\dot{\boldsymbol{u}}_{\boldsymbol{j}}(s)$, $\dot{\boldsymbol{p}}_{\boldsymbol{j}}(s)$ and $\dot{\boldsymbol{\varepsilon}}^p(\boldsymbol{y})$ should be found. They can be approximated by series with various base functions. In recent papers, the Lagrange polynomials were most often used for this purpose [3]. The number of expressions in the mentioned series depends on the number of assumed collocation (for displacements and tractions) or interpolation points (for plastic strains). The latter are arranged globally within the whole surface according to a predefined scheme e.g. uniformly or at places corresponding to roots of various kinds of polynomials. Using the trimming technique some of them can be outside the correct region (red points in Fig. 1e) and should not be used for the approximation of plastic strains. To determine them, a projection scheme is used [7]. For each interpolation point, the closest projection from it to the trimming curve

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is found. Then using the magnitude of the vector from the interpolation to the projection point and the tangential vectors of the trimming curve at the projection point the cross product is obtained. Taking into account the orientation rule, if its direction is coming out of the plane, the analyzed interpolation point is located outside of the correct region.

However, considering only some of interpolation points causes a problem with applying the Lagrange polynomials for approximation. It comes from the fact, that the set of points must have the same number of them in each row or column. Therefore, a method which allows for any distribution of interpolation points should be implemented. Many approaches can be applied [10], however, in this paper, the simplest is used - IDW method [9]. The formula for plastic strains approximation based on R points can be presented by

$$\dot{\boldsymbol{\varepsilon}}^p(\boldsymbol{y}) = \{ \begin{array}{c} \frac{\sum_{r=0}^{R} \omega_r(\boldsymbol{y}) \dot{\boldsymbol{\varepsilon}}^p(\boldsymbol{y}_r)}{\sum_{r=0}^{R} \omega_r(\boldsymbol{y})} \quad if \quad d(\boldsymbol{y}, \boldsymbol{y}_r) \neq 0 \quad for \quad all \quad r, \\ \dot{\boldsymbol{\varepsilon}}^p(\boldsymbol{y}_r) \quad if \quad d(\boldsymbol{y}, \boldsymbol{y}_r) = 0 \quad for \quad some \quad r, \end{array} \tag{2}$$

where $\omega_r(\boldsymbol{y}) = \frac{1}{d(\boldsymbol{y}, \boldsymbol{y}_r)^p}$ is a weighting function, d is a distance from the known $\boldsymbol{y}_r$ to to the unknown $\boldsymbol{y}$ point and p is a power parameter. To predict a value for any unmeasured location, IDW uses the measured values from a neighborhood of influence. In this paper it is determined by a circle with a radius r and the center at the unmeasured point.

The next step in solving PIES is to substitute the expression (2) into the formula (1) and to calculate the integrals, before applying the collocation method. The strategy used for boundary integrals is the same as for elastic problems, and the domain integrals are calculated over the whole surface using a higher-order quadrature [3]. Unfortunately, this time the strategy has to be modified since a part of the domain is trimmed. In elastoplastic problems, some singularities in domain integrals appear. The last integral from (1) is weakly singular, but this singularity can be canceled by employing the transformation technique [2, 3], in which the surface is divided into triangles at a point of singularity. The same technique is used for calculating strongly singular integrals in the stress integral identity [3]. It can be also, after some modifications, a direct solution to the problem of integration over the trimmed surface. Fig. 2 presents the way of division into triangles in the original version of PIES and with trimmed surfaces depending on the location of the singular point.

Fig. 2 shows that instead of using corner points of the surface, the real vertices of the considered body are used. The division takes place in the domain of the surface (unit square), therefore vertices should be recalculated to that parameter space. It can be done using formulas describing the surface by a point inversion algorithm, which in this paper is implemented only for the bilinear surface. For more complicated cases (e.g. trimming curves form a concave boundary), another strategy may be required. The initial geometry is divided into the smallest number of triangles using existing vertices. They are treated as separate surfaces and within them, the transformation technique is applied.

It should be emphasized, that the approximation formula (2) remains global over the trimmed surface. Moreover, if the trimming causes there are too few

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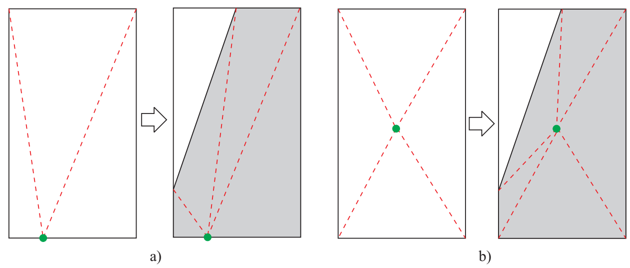

Fig. 2. The transformation technique with and without the trimmed surface: a) for collocation point, b) for interpolation point

interpolation points around the trimming curve, they can be easily generated according to the selected distribution along the given curve.

5 Examples

The first example is selected for initial verification as it has an analytical solution [11], although it could be solved without the trimming technique. The cantilever beam (Fig. 3a) is end-loaded, defined as plane stress with the material parameters: $E = 2 * 10^{11} Pa$, $\nu = 0.25$, $\sigma_0 = 20Pa$ and $H^{'} = 0$. The von Mises yield criterion is assumed (like in the remaining examples).

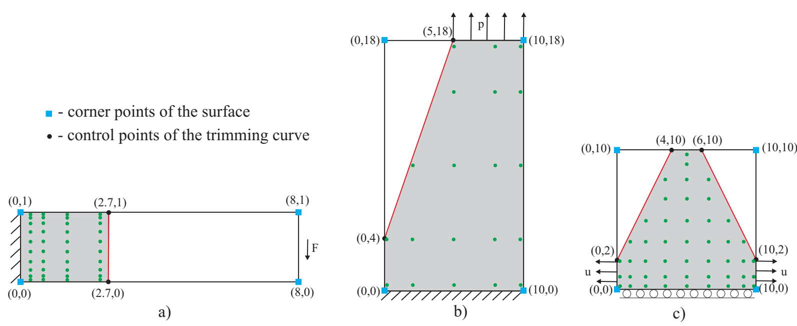

Fig. 3. The considered trimmed geometry for: a) first, b) second and c) third example

The whole considered domain is modeled by the bilinear surface and the assumed plastic zone (a part of that surface) is declared by the linear trimming curve. Initially, interpolation points (36,64,100) are arranged globally at the roots of Chebyshev polynomials of the first kind (this also applies to the remaining examples), but finally, the trimming curve and the projection scheme eliminate some of them (Fig. 3a). The area designated by the trimming curve is divided into triangles for calculating singular integrals (the same technique is used in

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the remaining examples). Plastic strains are approximated by the IDW method with $r = 0.4$, $p = 2$.

Tip deflection versus applied force is calculated using PIES with the various number of remaining interpolation points (R). The results are compared with analytical solutions and presented in Table 1 in the form of L^2 relative error norm. As can be seen, the error decreases as the number of interpolation points increases. Moreover, the analytically designated plastic zone overlaps with the area defined by yielded points obtained by PIES.

Table 1. L^2 error norm for tip deflection for various number of interpolation points

R	12	24	40
L^2	0.051991	0.045149	0.038378

The next considered geometry together with applied boundary conditions is presented in Fig. 3b. The domain of the problem is modeled by the bilinear surface and one linear trimming curve. The plane stress conditions with the following material properties are considered: $E = 70000MPa$, $\nu = 0.2$, $\sigma_0 = 150MPa$ and $H^{'} = 0$. Within the initial surface, 25 interpolation points are arranged. Some of them are finally not inside the plastic region, therefore they are eliminated by the projection scheme. The distribution of the remaining 20 interpolation points is shown in Fig. 3b. IDW is used with $r = 5$ and $p = 2$.

The displacements at the point $(10, 18)$ are obtained by PIES and FEM. The load-displacement curve is presented in Fig. 4. It demonstrates the good agreement between analyzed solutions. Taking FEM results as reference (due to the lack of analytical solution), L^2 relative norm of displacements is 0.0214231. There is also agreement in the yield area between both tested methods. Moreover, PIES is much less computationally demanding: in FEM 484 equations were solved, while in PIES only 80.

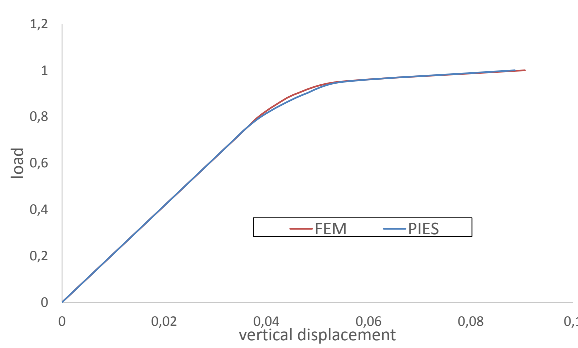

Fig. 4. The load-displacement curve

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The last example concerns the geometry under the imposed displacement modeled using the surface trimmed by two curves (Fig. 3c). The perfect plasticity under plane stress state with $E = 1MPa$, $\nu = 0.3$, $\sigma_0 = 0.9MPa$ is assumed. 47 of the globally placed points remain in the correct area. The IDW parameters are $r = 5$, $p = 2$.

The von Mises equivalent stresses σ_{VM} along the cross-section $x = 5$ are obtained. Due to the lack of analytical solution, they are compared with FEM results and L^2 norm is calculated. It equals 0.0229775 which means that the results are very similar, but computational requirements are much smaller for PIES than for FEM (PIES 84, FEM 586 equations).

6 Conclusions

The paper presents PIES with trimmed surfaces for solving elastoplastic problems. The domain is modeled by a surface and its unnecessary part is trimmed by curves. The approximation of the plastic strains is performed by the IDW method, which allows for any arrangement of interpolation points. Finally, the domain integrals are calculated over the trimmed surface using the modified division technique that cancels out the singularity in the integrand. In more complicated cases, the earlier division into multiple surfaces may be required.

The proposed approach is tested on three examples with various numbers of trimming curves. The obtained results are in good agreement with analytical and FEM solutions. Further research is required, on more complex examples, e.g. with holes or a curved boundary.

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