Interval modification of the fast PIES in solving 2D potential BVPs with uncertainly defined polygonal boundary shape

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Abstract. The paper presents a new modification of the fast parametric integral equations system (FPIES) by application of interval numbers and interval arithmetic in solving potential 2D boundary value problems with complex shapes. Obtained interval modified fast PIES is used to model the uncertainty of measurement data, which are necessary to define boundary shape. The uncertainty was defined using interval numbers and modelled using modified directed interval arithmetic previously developed by the authors. The reliability and efficiency of the interval modified fast PIES solutions obtained using such arithmetic were verified on 2D complex potential problems with polygonal domains. The solutions were compared with the interval solutions obtained by the interval PIES. All performed tests indicated high efficiency of the interval modified fast PIES method.

Keywords: Fast PIES · Interval numbers · Interval arithmetic · Directed intervals

1 Introduction

One of the robust numerical tools for solving boundary value problems (BVPs) is the parametric integral equations system (PIES) [1]. The method was successfully used to solve many different problems (e.g. [2, 3]). The disadvantages of the PIES connected with the generation of dense non-symmetric coefficient matrices and with the method of solving the final system of algebraic equations (Gaussian elimination) were fixed by the application of the fast multipole method (FMM) [4]. Obtained fast PIES (FPIES) [5, 6] significantly reduced the computation time, as well as the problem of huge random access memory (RAM) utilization.

The authors of this paper also developed the interval PIES (IPIES) [7], which is used to solve uncertainly defined problems. It is known, that in modelling and solving BVPs the shape of the boundary, boundary conditions and some parameters of the considered domain (i.e. material properties) should be defined. In practice, to obtain these data we should measure some physical quantities.

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However, even the most precise measurement is not exact - inaccuracy of measurement instruments, gauge reading error or approximations of the models used in the analysis of measurements affect the accuracy of determining the physical quantity.

It should be noted, that the direct consideration of uncertainty in classical mathematical models is not possible - they required exact values of the data. However, in the literature, we can find a lot of modifications of known methods considered uncertainty (e.g. [8–10]). One of them is connected with the application of interval numbers and interval arithmetic to the method of modelling and solving uncertainly defined BVPs. Therefore, it was used in the interval finite element method (IFEM)[11] and the interval boundary element method (IBEM)[12], as well as the IPIES.

In general, either in IFEM or IBEM the uncertainty of the boundary shape is not considered (only material parameters or boundary conditions). Only in a few papers, some parameters of the shape (such as radius or beam length) were uncertainly defined. In the IPIES all uncertainties can be considered simultaneously [13]. Although the IPIES has advantages inherited from the PIES, such as the way of defining the boundary connected with a small number of interval control points, there are also some disadvantages. Unfortunately, the application of interval arithmetic and interval numbers made computations slower and utilized more RAM than in the PIES. Therefore, solving complex (large-scale) uncertainly defined problems required a combination of the IPIES and the FPIES.

The main goal of this paper is to present the interval modified fast PIES (IFPIES) applied for numerical solving of 2D potential complex BVPs with uncertainly defined boundary shapes. The application of interval arithmetic and interval numbers into the FPIES was required to obtain the new method for modelling and solving uncertainly defined problems. The efficiency and accuracy of the IFPIES are tested on the potential problems with polygonal domains.

2 Modelling uncertainty of the boundary shape

Direct application of either classical [14] or directed [15] interval arithmetic for modelling boundary problems with uncertainly defined boundary shapes is very troublesome as presented in [13]. The main problem is the consideration of unrealistic problems as a result of the lack of continuity between boundary segments. Modelling the same boundary shape in different quadrants of the Cartesian coordinate system gives different results. Therefore, the authors proposed to modify the directed interval arithmetic by mapping arithmetic operators to the positive semi-axis as clearly described in [13].

In this paper, for modeling uncertainly defined boundary shape, linear segments in form of interval Bézier curves of the first degree are used:

$$\boldsymbol{S}_k(s) = (1-s)\boldsymbol{P}_{b(k)} + s\boldsymbol{P}_{e(k)}, \quad 0 \le s \le 1 \tag{1}$$

where $\boldsymbol{S}_k(s) = \{\boldsymbol{S}_k^{(1)}(s), \boldsymbol{S}_k^{(2)}(s)\}$, $k = \{1, 2, ..., n\}$ - the number of segments created boundary, s - variable in parametric reference system, $^{(1)}$ and $^{(2)}$ - the

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direction of coordinates in 2D Cartesian reference system, $\boldsymbol{P}_{b(k)}$,$\boldsymbol{P}_{e(k)}$ - interval endpoints which define interval Bézier curves as presented in Fig. 1.

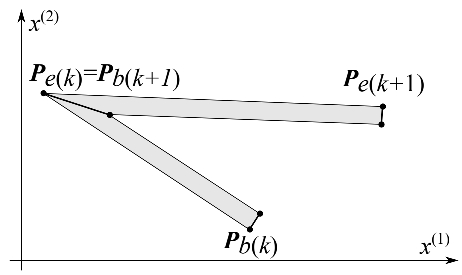

Fig. 1. The interval Bézier curve of the first degree used to define a segment of the boundary in the IPIES

3 The interval modified fast PIES (IFPIES)

The FPIES for 2D potential problems [5] was obtained as the result of modification of the PIES. It includes the modification of the PIES kernels to allow for the Taylor series approximation used by the FMM. Also, the tree used by the FMM was modified to properly include the way of defining the boundary in the PIES [16]. IFPIES is obtained similarly to the FPIES. However, the application of the modified directed interval arithmetic and interval numbers is not trivial. Some variables should be defined as complex intervals, i.e. either real or imaginary part of a complex number is treated as an interval.

The basic form of the IFPIES formula is similar to the FPIES [5], however most variables are defined using interval numbers similarly to the IPIES [7]:

$$\begin{aligned} \frac{1}{2}u_l(\widehat{s}) &= \sum_{j=1}^{n} \mathbb{R}\left\{ \int_{s_{j-1}}^{s_j} \widehat{\boldsymbol{U}}_{lj}^{*(c)}(\widehat{s},s)p_j(s)\boldsymbol{J}_j^{(c)}(s)ds \right\} - \\ &- \sum_{j=1}^{n} \mathbb{R}\left\{ \int_{s_{j-1}}^{s_j} \widehat{\boldsymbol{P}}_{lj}^{*(c)}(\widehat{s},s)u_j(s)\boldsymbol{J}_j^{(c)}(s)ds \right\}, \\ & l=1,2,...,n,\ s_{l-1} \leq \widehat{s} \leq s_l,\ s_{j-1} \leq s \leq s_j, \end{aligned} \tag{2}$$

where: $\widehat{s}$ and s are defined exactly in the parametric coordinate system, s_{j-1} (s_{l-1}) correspond to the beginning and s_j (s_l) to the end of interval segment $\boldsymbol{S}_j$ ($\boldsymbol{S}_l$), n is the number of parametric segments that creates boundary of domain in 2D, $\widehat{\boldsymbol{U}}_{lj}^{*(c)}(\widehat{s},s)$ and $\widehat{\boldsymbol{P}}_{lj}^{*(c)}(\widehat{s},s)$ are interval kernels, $\boldsymbol{J}_j^{(c)}(s)$ is the interval Jacobian, $u_j(s)$ and $p_j(s)$ are parametric boundary functions on individual segments $\boldsymbol{S}_j$ of the interval boundary, $\mathbb{R}$ is the real part of complex function.

In the IFPIES integrals are computed using the same formulas as in the FPIES (clearly derived in [5, 6]). The main difference is in the way of defining

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interval variables. At last, the IFPIES integrals are described as follows:

$$\begin{aligned}
&\int\limits_{s_{j-1}}^{s_j} \widehat{\boldsymbol{U}}_{lj}^{*(c)}\left(\widehat{s},s\right)p_j\left(s\right)\boldsymbol{J}_j^{(c)}\left(s\right)ds=\frac{1}{2\pi}\sum_{l=0}^{N_T}(-1)^l\cdot\\
&\quad\cdot\left\{\sum_{k=0}^{N_T}\sum_{m=l}^{N_T}\frac{(k+m-1)!\cdot\boldsymbol{M}_k(\boldsymbol{\tau}_c)}{\left(\boldsymbol{\tau}_{el}-\boldsymbol{\tau}_c\right)^{k+m}}\cdot\frac{\left(\boldsymbol{\tau}'_{el}-\boldsymbol{\tau}_{el}\right)^{m-l}}{(m-l)!}\right\}\frac{\left(\widehat{\boldsymbol{\tau}}-\boldsymbol{\tau}'_{el}\right)^l}{l!},\\
&\int\limits_{s_{j-1}}^{s_j} \widehat{\boldsymbol{P}}_{lj}^{*(c)}\left(\widehat{s},s\right)u_j\left(s\right)\boldsymbol{J}_j^{(c)}\left(s\right)ds=\frac{1}{2\pi}\sum_{l=0}^{N_T}(-1)^l\cdot\\
&\quad\cdot\left\{\sum_{k=1}^{N_T}\sum_{m=l}^{N_T}\frac{(k+m-1)!\cdot\boldsymbol{N}_k(\boldsymbol{\tau}_c)}{\left(\boldsymbol{\tau}_{el}-\boldsymbol{\tau}_c\right)^{k+m}}\cdot\frac{\left(\boldsymbol{\tau}'_{el}-\boldsymbol{\tau}_{el}\right)^{m-l}}{(m-l)!}\right\}\frac{\left(\widehat{\boldsymbol{\tau}}-\boldsymbol{\tau}'_{el}\right)^l}{l!}.
\end{aligned} \tag{3}$$

where: N_T is the number of of terms in the Taylor expansion, $\widehat{\boldsymbol{\tau}}=\boldsymbol{S}_l^{(1)}(\widehat{s})+i\boldsymbol{S}_l^{(2)}(\widehat{s})$, $\boldsymbol{\tau}=\boldsymbol{S}_j^{(1)}(s)+i\boldsymbol{S}_j^{(2)}(s)$, complex interval points $\boldsymbol{\tau}_c$, $\boldsymbol{\tau}_{el}$, $\boldsymbol{\tau}'_c$, $\boldsymbol{\tau}'_{el}$ are mid-points of leaves obtained during tracing the tree structure (see [5, 16]). Expressions $\boldsymbol{M}_k(\boldsymbol{\tau}_c)$ and $\boldsymbol{N}_k(\boldsymbol{\tau}_c)$ are called moments (and they are computed twice only) and have the form [5, 16]:

$$\begin{aligned}
\boldsymbol{M}_k(\boldsymbol{\tau}_c)&=\int\limits_{s_{j-1}}^{s_j}\frac{\left(\boldsymbol{\tau}-\boldsymbol{\tau}_c\right)^k}{k!}p_j\left(s\right)\boldsymbol{J}_j^{(c)}\left(s\right)ds,\\
\boldsymbol{N}_k\left(\boldsymbol{\tau}_c\right)&=\int\limits_{s_{j-1}}^{s_j}\frac{\left(\boldsymbol{\tau}-\boldsymbol{\tau}_c\right)^{k-1}}{(k-1)!}\boldsymbol{n}_j^{(c)}u_j\left(s\right)\boldsymbol{J}_j^{(c)}\left(s\right)ds.
\end{aligned} \tag{4}$$

where $\boldsymbol{n}_j^{(c)}=\boldsymbol{n}_j^{(1)}+i\boldsymbol{n}_j^{(2)}$ the complex interval normal vector to the curve created segment j.

The IPIES is solved using the pseudospectral method, therefore it is written at collocation points whose number corresponds to the number of unknowns (described in [1]). Hence, obtained interval system of algebraic equations can be compact written as $\boldsymbol{H}\mathbf{u}=\boldsymbol{G}\mathbf{p}$, where $\mathbf{u}$ and $\mathbf{p}$ are column vectors containing coefficients of approximating boundary functions $u_j(s)$ and $p_j(s)$ respectively. This system is transformed into the system of interval algebraic equations $\boldsymbol{A}\cdot\boldsymbol{x}=\boldsymbol{b}$ depending on the given type of boundary conditions. The vector $\boldsymbol{x}$ represents unknown coefficients and the column vector $\boldsymbol{b}$ contains given boundary conditions. The matrix $\boldsymbol{A}$ is dense, therefore in the IPIES direct solver in form of interval Gaussian elimination was used to solve the system.

Unlike the IPIES, the IFPIES produces the system of algebraic equations implicitly, i.e. only the result of multiplication of the matrix $\boldsymbol{A}$ by the vector of unknowns $\boldsymbol{x}$ is obtained. Therefore, an iterative GMRES solver modified by the application of directed interval arithmetic directly integrated with the FMM was applied in the IFPIES. However, the direct solver in the IPIES requires $O(N^3)$

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operations to solve the interval system of algebraic equations (N is the number of equations). We also applied the GMRES solver to the IPIES to obtain a more reliable comparison.

4 Numerical results

The first example is the gear-shaped plate presented in the Fig. 2a. The problem is described by Laplace's equation. The boundary contains 1 024 segments. Boundary conditions are also presented in Fig. 2a (where u - Dirichlet and p - Neumann boundary conditions). Tests are performed on a PC based on Intel Core i5-4590S with 16 GB RAM. Application of the IPIES and the IFPIES are compiled by g++ 7.5.0 (-O2 optimization) on 64-bit Ubuntu Linux OS (kernel 5.4.0).

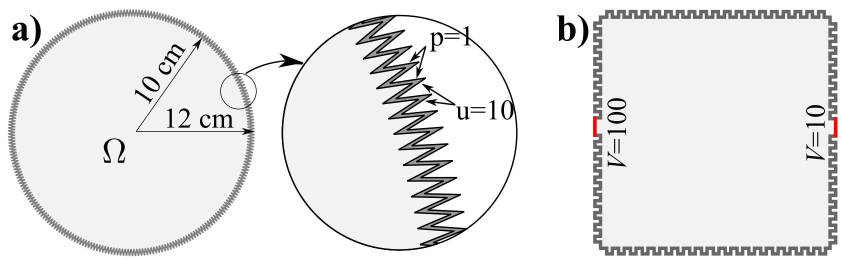

Fig. 2. Considered a) the gear-shaped, b) the square shaped boundary problem

The research focused on the CPU time, RAM utilization and the accuracy of the IFPIES compared to the IPIES. The mean square error (MSE) between infimum and supremum of the IFPIES and the IPIES solutions are computed to prove the accuracy of the proposed method.

Table 1. Comparison between the IFPIES and the IPIES

Number of		CPU time [s]		RAM utilization [MB]		MSE	
col. pts	eqs	*IFPIES*	*IPIES*	*IFPIES*	*IPIES*	inf	sup
2	2 048	12.09	37.07	28.28	197	0.0	0.0
4	4 096	49.78	160.65	83.81	775	$1.05 \cdot 10^{-13}$	$8.11 \cdot 10^{-14}$
6	6 144	119.05	395.83	168	1 741	$2.62 \cdot 10^{-10}$	$2.73 \cdot 10^{-10}$
8	8 192	239.29	768.29	316	3 122	$2.07 \cdot 10^{-11}$	$7.34 \cdot 10^{-11}$

Approximation of the modified PIES kernels uses 25 terms in the Taylor series, and the GMRES tolerance is equal to 10^{-8}. The number of collocation points is the same on all segments and equal to 2, 4, 6 or 8. Therefore, we should solve the system of 2 048, 4 096, 6 144 and 8 192 equations respectively.

As can be seen from Tab. 1, the IFPIES is about 3 times faster and uses up to 10 times less RAM than the IPIES. However, the mean square error (MSE) between both methods is on a very low level and does not exceed 10^{-10}. Hence, the IFPIES is as accurate as the IPIES.

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The second example is the current flow through a square plate presented in Fig. 2b. The problem is also described by Laplace's equation. The boundary is composed of 16 004 segments. Potentials V (Dirichlet boundary conditions) are applied to two electrodes presented in Fig. 2b. Neumann boundary conditions in the rest of the boundary are equal to 0.

As in the previous example, 25 terms in the Taylor series, the GMRES tolerance equal to 10^{-8} and the number of collocation points from 2 to 8 are applied (the system of 32 008 to 128 032 equations is solved).

This example cannot be solved by the IPIES on a standard PC due to very high RAM utilization. Therefore, computations were carried out at the Computer Center of the University of Bialystok on Intel Xeon E5-2650v2 with 512 GB RAM. Application of the IPIES and the IFPIES are compiled by g++ 8.3.0 (-O2 optimization) on 64-bit OpenHPC and Centos Linux OS (kernel 3.10.0).

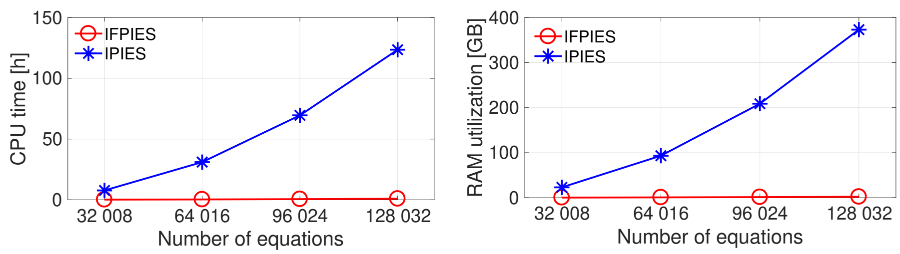

Fig. 3. The CPU time and the RAM utilization of the problem of the current flow through a square plate

As can be seen from a Fig. 3 relationship between the CPU time (the RAM utilization) and the number of equations in the IFPIES is close to linear contrary to the IPIES. The IFPIES uses about 53 min and 2.2 GB RAM contrary to 123.5 h and 373.2 GB RAM in the IPIES for the example with 128 032 equations. Hence, the IFPIES allows for solving large-scale uncertainly defined problems in a reasonable time and small RAM utilization on a standard PC.

5 Conclusions

The paper presents the IFPIES in solving 2D potential uncertainly defined boundary value problems. The FPIES was previously applied in modelling and solving 2D single- and multi-connected certainly defined potential problems. Applied fast multipole technique with a modified binary tree allows for significant reduction of CPU time, as well as RAM utilization. Also, the IFPIES allows for highly efficient solving of complex engineering problems on a standard PC in a reasonable time. However, the real power of the IFPIES is connected with the large size of solved problems and low RAM utilization. The IPIES allows solving the problems with a system up to about 25 000 equations on a standard PC with 16 GB of RAM, whilst 128 032 equations in the IFPIES use about 2.2 GB of RAM only.

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Obtained results strongly suggest that the direction of research should be continued. The authors want to extend the algorithm of the IFPIES to problems with curvilinear boundary shapes, as well as modelled by other equations.

References

1. E. Zieniuk, Hermite curves in the modification of integral equations for potential boundary-value problems, Engineering Computations 20(1-2), 112-128 (2003).
2. A. Kużelewski, E. Zieniuk, OpenMP for 3D potential boundary value problems solved by PIES, in 13th International Conference of Numerical Analysis and Applied Mathematics ICNAAM 2015, AIP Conf. Proc. 1738, 480098 (2016).
3. A. Kużelewski, E. Zieniuk, A. Bołtuć, Application of CUDA for Acceleration of Calculations in Boundary Value Problems Solving Using PIES, in: Parallel Processing and Applied Mathematics PPAM 2013, LNCS 8385, Part II, Springer-Verlag, Berlin, 322-331 (2014).
4. L.F. Greengard, V. Rokhlin, A fast algorithm for particle simulations, Journal of Computational Physics 73(2), 325-348 (1987).
5. A. Kużelewski, E. Zieniuk, The fast parametric integral equations system in an acceleration of solving polygonal potential boundary value problems. Advances in Engineering Software 141, 102770 (2020).
6. A. Kużelewski, E. Zieniuk, Solving of multi-connected curvilinear boundary value problems by the fast PIES, Computer Methods in Applied Mechanics and Engineering 391, 114618 (2022).
7. E. Zieniuk, M. Kapturczak, A. Kużelewski, Modification of Interval Arithmetic for Modelling and Solving Uncertainly Defined Problems by Interval Parametric Integral Equations System, In: International Conference on Computational Science ICCS 2018, LNCS 10862, Springer-Verlag, Berlin, 231-240 (2018).
8. Z. Gouyandeh, T. Allahviranloo, S. Abbasbandy, A. Armand, A fuzzy solution of heat equation under generalized Hukuhara differentiability by fuzzy Fourier transform. Fuzzy Sets and Systems 309, 81-97 (2017).
9. C. Wang, H.G. Matthies, Dual-stage uncertainty modeling and evaluation for transient temperature effect on structural vibration property. Computational Mechanics 63, 323-333 (2019).
10. C. Fu, Q. Zhan, W. Liu, Evidential reasoning based ensemble classifier for uncertain imbalanced data. Information Sciences 578, 378-400 (2021).
11. B.Y. Ni, C. Jiang, Interval field model and interval finite element analysis. Computer Methods in Applied Mechanics and Engineering 360, 112713 (2020).
12. B. Zalewski, R. Mullen, R. Muhanna, Interval boundary element method in the presence of uncertain boundary conditions, integration errors, and truncation errors. Engineering Analysis with Boundary Elements 33, 508-513 (2009).
13. E. Zieniuk, M. Czupryna, The strategy of modeling and solving the problems described by Laplace's equation with uncertainly defined boundary shape and boundary conditions, Information Sciences 582, 439-461 (2022).
14. R.E. Moore, Interval Analysis. Prentice-Hall, Englewood Cliffs, New York (1966).
15. S.M. Markov, On directed interval arithmetic and its applications. Journal of Universal Computer Science 1 (7), 514-526 (1995).
16. A. Kużelewski, E. Zieniuk, A. Bołtuć, K. Szerszeń, Modified binary tree in the fast PIES for 2D problems with complex shapes, in: International Conference on Computational Science ICCS 2020, LNCS 12138, Part II, Springer-Verlag, Berlin, 1-14 (2020).

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