Simple and Efficient Acceleration of the Smallest Enclosing Ball for Large Data Sets in E^2: Analysis and Comparative Results ⋆

Vaclav Skala[0000−0001−8886−4281], Matej Cerny, and Josef Yassin Saleh

University of West Bohemia, Faculty of Applied Sciences
Dept. of Computer Science and Engineering
Pilsen, CZ 301 00, Czech Republic
skala@kiv.zcu.cz matcerny@kiv.zcu.cz salehj@kiv.zcu.cz
www.VaclavSkala.eu

Abstract. Finding the smallest enclosing circle of the given points in E^2 is a seemingly simple problem. However, already proposed algorithms have high memory requirements or require special solutions due to the great recursion depth or high computational complexity unacceptable for large data sets, etc. This paper presents a simple and efficient method with speed-up over 100 times based on processed data reduction. It is based on efficient preprocessing, which significantly reduces points used in final processing It also significantly reduces the depth of recursion and memory requirements, which is a limiting factor for large data processing. The proposed algorithm is easy to implement and it is extensible to the E^3 case, too. The proposed algorithm was tested for up to 10^9 of points using the Halton's and "Salt and Pepper" distributions.

Keywords: Smallest enclosing circle · smallest enclosing ball · algorithm complexity · preprocessing · convex hull · convex hull diameter.

1 Introduction

Algorithms for finding the smallest enclosing circle in the E^2 case, or the enclosing ball in the E^k general case, have been studied for a long time and many algorithms have been published with many modifications. Sylvester[62] made the first problem formulation in 1857 and later by others, see Elzinga[10]. Several algorithms have been published, e.g. Megiddo's algorithm[31] with an overview of some other interesting solutions, Ritter[46], etc. A brief introduction to the problem is available at WiKi[71][72].

An interesting approach was published by Welzl[67] in 1991. It is a "brute force" recursive algorithm with a random selection of points. It leads to a significant speed-up due to random point selection. However, it is not directly usable for large data sets due to the very deep recursion calls.

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Unfortunately, the originally proposed Welzl's algorithm is partially incorrect and Matoušek, Sharir, Welzl's published the corrected version, as the MSW algorithm[30] (the code available on WiKi[72]), see[70][72].

Algorithm 1 MSW - Matousek, Sharir, Welzl's algorithm

```
Require: Finite sets P and R of points in the plane |R| = 3
Ensure: Minimal disk enclosing P ∪ R
  if P is empty then
      return trivial(R)
  end if
  choose p in P                                        ▷ randomly and uniformly
  D := msw(P − {p}, R)
  if p is in P then
      return(D)
  end if
  q = nonbase(R ∪ {p})
        ▷ Welzl's algorithm for 4 points could be used to find what would not be in R
  return MSW (P − {p} ∪ {q}, R ∪ {p} − {q})
```

It should be noted, that there is no significant difference between the original Welzl's and MSW algorithms as far as the timing is concerned.

2 Proposed preprocessing

The smallest enclosing center problem is closely related to the diameter of the convex hull problem. A simple algorithm with the $O_{exp}(N)$ complexity for finding a diameter of the convex hull of points using preprocessing was published by Skala[53][55][58]. The algorithm based on polar space subdivision was introduced in Skala,Smolik,Majdisova[59] and extended in Skala,Majdisova,Smolik[57] for the E^3 case.

2.1 Convex hull diameter estimation

It is based on a simple idea. The AABB points and points closest to the AABB corners form a convex hull. The maximum distance d defines an estimation of the convex hull diameter, i.e. radius r. Then the given points Ω are split to $\Omega_0, \ldots, \Omega_4$, see Fig.1.

It can be seen that the Ω_0 points cannot contribute to the convex hull diameter. Then points of Ω_1 and Ω_3 are processed and the value r is updated. Similarly, updates of the radius r are made after Ω_2 and Ω_4, Ω_1 and Ω_2, Ω_2 and Ω_3, Ω_3 and Ω_4, Ω_4 and Ω_1. This leads to significant reduction of points that could form the final convex hull. Then the final diameter of the convex hull is computed; see Skala[53][56][58] for details.

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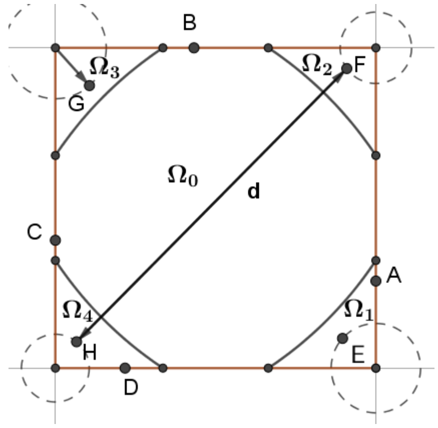

Fig. 1: Maximum distance estimation

Other efficient algorithms for finding the convex hull of points in the E^2 case were published in Skala[55][59] and for the convex hull in E^3 case was described in Skala[57]. This preprocessing leads to significant speed up, see Skala[58] for details. However, the polar subdivision used in Smolik[61] is quite complex to implement.

The Welzl's recursive algorithm is based on the "brute force" approach actually, but the randomized point heuristic selection use leads to the $O_{exp}(N)$ expected complexity, where N is a number of points. Unfortunately, it leads to deep recursive calls, which is a very limiting factor for large data processing.

2.2 Theoretical analysis

The simplest acceleration of the smallest enclosing circle algorithm is to find points that form the Axis Aligned Bounding Box (AABB), i.e. points A, B, C, D. The worst case is when the AABB is a square and the points A, B, C, D are at the middle of the edges, see Fig.2a. In this case, all points inside of the area Ω_0 can be removed from the future processing. However, if points closest to the AABB corners are found, i.e. points E, F, G, H, all points inside of the convex polygon $A, \ldots, H$ can be removed. The points E, F, G, H are on an expected distance r from corners and the radius r decreases with the number N of the given points.

The Fig.2a presents a general case with a rectangular area. The closest point to a corner of the AABB lies on a circle with the expected radius r. It should be noted that only $\frac{1}{4}$ of the circle area is inside of the AABB. The radius r depends on the number of the given points N. If the regular orthogonal distribution of points in the E^2 case forms a mesh of $\sqrt{N} \times \sqrt{N}$ points. However, in the following a uniform distribution of points is expected and the "Salt and Pepper" Chen[5] and Halton's[71] and distributions were used in experiments.

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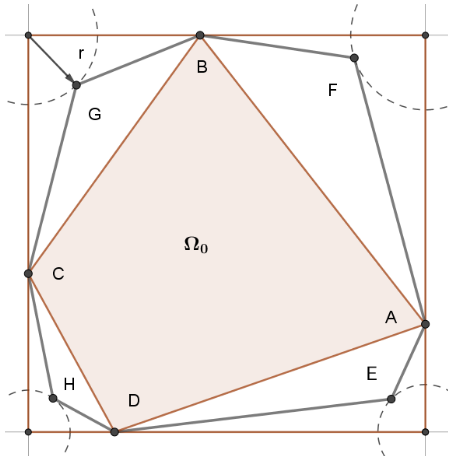
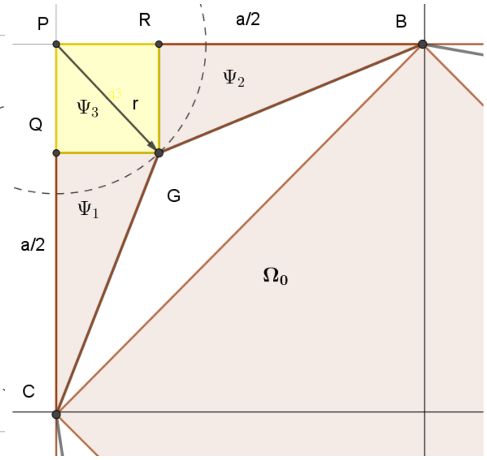


(a) Convex hull estimation



(b) Corner area

Fig. 2: Axis Aligned Bounding Box and convex polygon

Let P_0 is the area of the square with edges of the length a. The P_1 is the area of the circular sector of the radius r, see Fig.2a, containing just one point.

$$\frac{1}{N} = \frac{P_1}{P_0} \qquad pN = 1 \qquad p = \frac{P_1}{P_0} \qquad N\frac{\frac{1}{4}\pi r^2}{a^2} = 1 \qquad r^2 = \frac{4a^2}{\pi N} \tag{1}$$

Then in the E^2 case the expected radius r can be estimated as:

$$r = \frac{2a}{\sqrt{\pi}\sqrt{N}} = \frac{2}{\sqrt{\pi}}\frac{a}{\sqrt{N}} \approx 1.1284\frac{a}{\sqrt{N}} \tag{2}$$

The sizes of the corner's areas $\Psi_1 + \Psi_2 + \Psi_3$ can be estimated as:

$$P_2 = \frac{a}{2}\frac{r}{\sqrt{N}} \tag{3}$$

It means, that the P_2 area decreases with the value N significantly. If the total number of points is $N = 10^8$, i.e. $\sqrt{N} = 10^4$, then

$$r \approx 1.1284\frac{a}{10^4} = 0.00011284\ a \qquad P_2 \approx \frac{0.00011284}{2\sqrt{N}}\ a^2 \tag{4}$$

It can be seen, that the estimated point reduction is very high. In the E^3 case, the number of points in the given set is N and only $\frac{1}{8}$ of the corner ball volume are inside of the AABB. The expected radius r can be estimated as:

$$pN = 1 \qquad p = \frac{V_1}{V_0} \qquad N\frac{\frac{1}{8}\frac{4}{3}\pi r^3}{a^3} = 1 \qquad r^3 = \frac{6a^3}{\pi\sqrt[3]{N}} \tag{5}$$

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In the E^3 case, the expected radius r can be estimated as:

$$r = \frac{\sqrt[3]{6}\ a}{\sqrt[3]{\pi}\sqrt[3]{N}} \approx 1.2407 \frac{a}{\sqrt[3]{N}} \tag{6}$$

This gives some estimation of the efficiency of the proposed preprocesing.

2.3 Implementation notes

The algorithm for finding the convex polygon $A, \ldots, H$ consists of four passes with $O(N)$ complexity (N is the total number of points). However, it should be noted that after the second step a significant fraction of points is discarded already.

Algorithm 2 Smallest enclosing circle algorithm

Require: All given points in the $\mathbf{\Omega}$ set
Ensure: The smallest enclosing circle
FIND the points forming the AABB, i.e. $A, \ldots, D$.
SPLIT the points into four areas half-planes based on their position relative to the line segments formed by the points $A, \ldots, D$.
▷ points that do not fit into any of the half-planes, i.e. quad
▷ points inside of $ABCD$ quad, are promptly discarded
FIND points $E, \ldots, H$. ▷ distance of the points is measured only to the corner
▷ of their respective half-plane
REMOVE the remaining points inside of the convex polygon $A, \ldots, H$
▷ information about point's half-plane is used to reduce further testing
▷ the points in $\mathbf{\Omega_0}$ are removed as they cannot influence the final smallest ball
CALL the Welzl's algorithm [MSW] for the remaining points ▷ see Fig.5a

This approach of discarding the significant fraction of points at the beginning has proven to be superior to the simple point-in-polygon test, as such test needs to find the whole polygon first, which, among other things, requires measuring distance to all four AABB corners per point. [1]
Further reduction of read/write operations can be achieved by using separate data structure for storing the index of a region Ω_i for a given point.

3 Experimental results

The proposed modification of the Welzl's algorithm was tested for a large number of points (up to $N > 4 * 10^8$ points) in the E^2 case. The Halton's and "Salt and Pepper" distributions were used for experiments. Experiments proved the following expected properties:

[1] Also, instead of computing the distance between points d, the $\sqrt{d^2}$ should not be used and d^2 can be used for distance comparisons Skala[52][54]. Same idea can be applied to the radius of a circumscribed circle in Welzl's algorithm.

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- the proposed smallest enclosing circle with preprocessing algorithm is of the $O_{exp}(N)$ time complexity even for large data sets (the Halton's and "Salt and Pepper" distributions used),
- timing and significant speed up due to preprocessing, see Fig.3a and Fig.3b respectively,
- the reduction ratio grows with the number of points $O\sqrt{N}$, see Fig.5a,
- the relative processing time $time/N$ is nearly constant for $N \geq 10^4$, see Fig.4b,
- significant decrease in the recursion depth as the result of preprocessing, which in turn leads to higher memory efficiency.

Implementation was done in C++, x64, compiler MSVC, Windows 10, 16GB RAM, Intel i7-10750H, 2.60GHz, 6 Cores CPU.
One of the main advantages of the proposed preprocessing is the significant reduction of the recursion depth. It can be seen that the depth recursion required is nearly 10^4 less if the proposed preprocessing is used, see Fig.5b for $N = 10^8$ points. There is also a direct influence on the computational time required.

4 Conclusion

The proposed algorithm for the acceleration of finding the smallest enclosing ball is simple, fast, robust and easy to implement. The significant advantages over recent solutions are:

- significant speed-up up to 10^2 times and it grows with the number of points,
- significant reduction of the depth of recursion, which is a limiting factor of the original algorithm for large data sets processing,
- significant data reduction before the final Welzl's (MSW) algorithm use,
- simple extensibility of the preprocessing algorithm to the E^3 case,
- better memory management, caching, during data processing,

The Halton's and "Salt and Pepper" distributions were used and the experimental results proved the speed-up expected. However, there is a potential for additional speed-up using SIMD instructions or GPU, use of the more advanced algorithms, e.g. the $O(\lg N)$ or $O(1)$ point in the convex polygon algorithms Skala[51], circumscribed sphere algorithm Skala[54]. [2]

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[2] Responsibilities: Algorithm design and analysis, manuscript preparation - V.Skala, Experimental implementation and verification - M.Cerny and J.Y. Saleh

[3] SIMD version using Intel's intrinsics AVX-2 has been tested and led to an additional roughly 20% performance gain.

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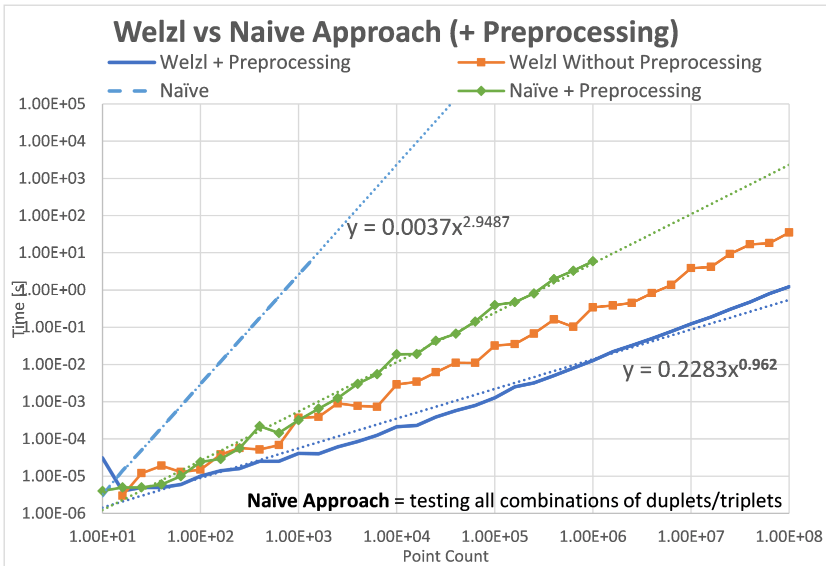
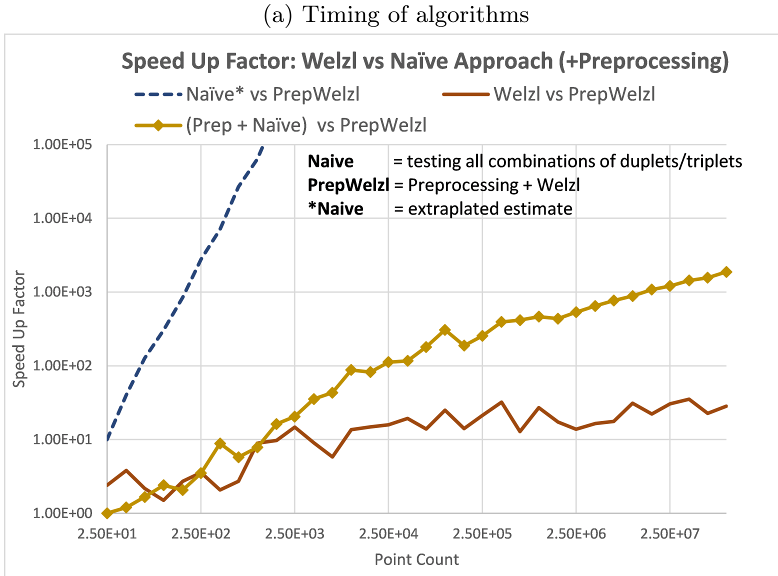


(a) Timing of algorithms



(b) The corner area influence

Fig. 3: Timing and corner areas influence

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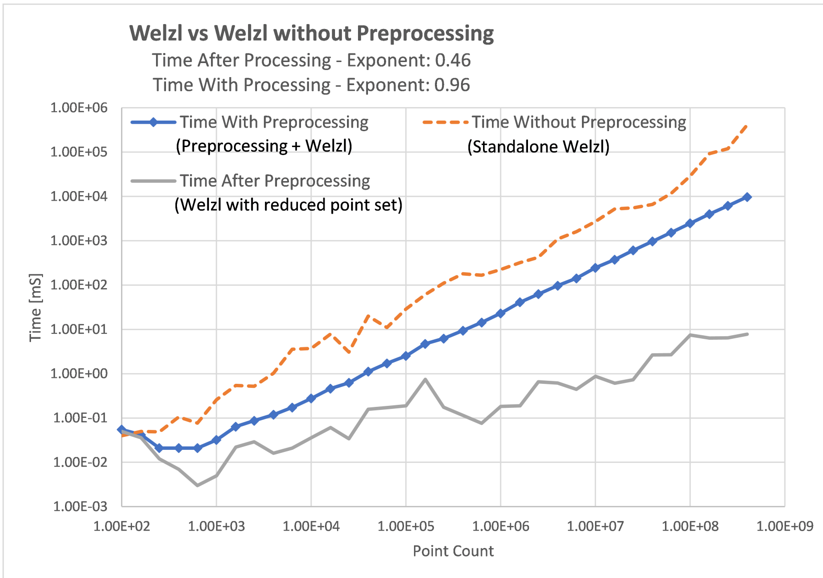
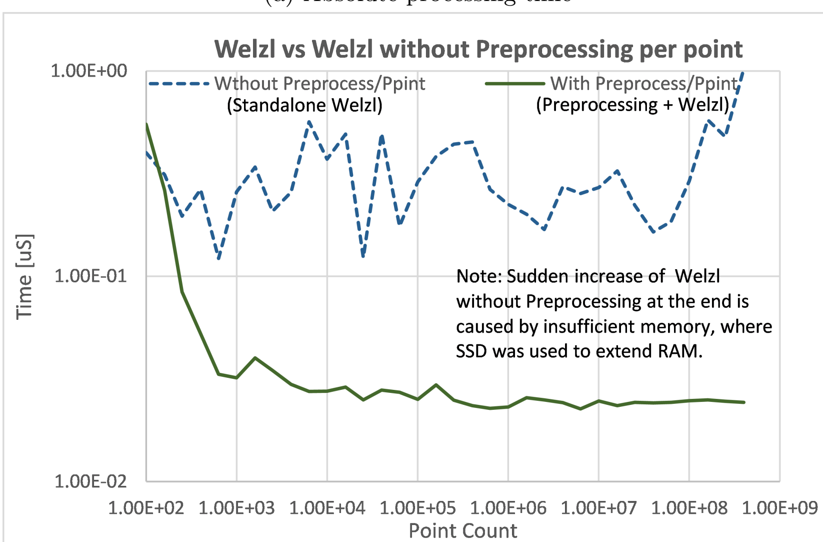


(a) Absolute processing time



(b) Relative processing time, i.e. $\frac{time}{N}$

Fig. 4: Absolute and relative processing times

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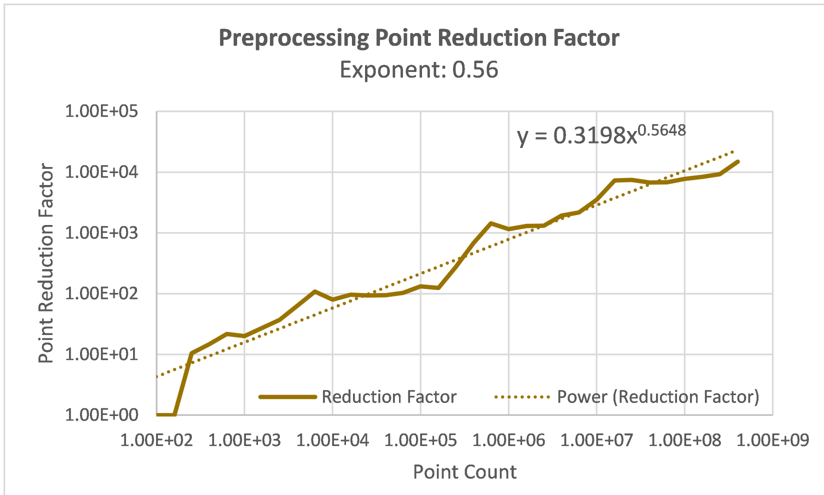
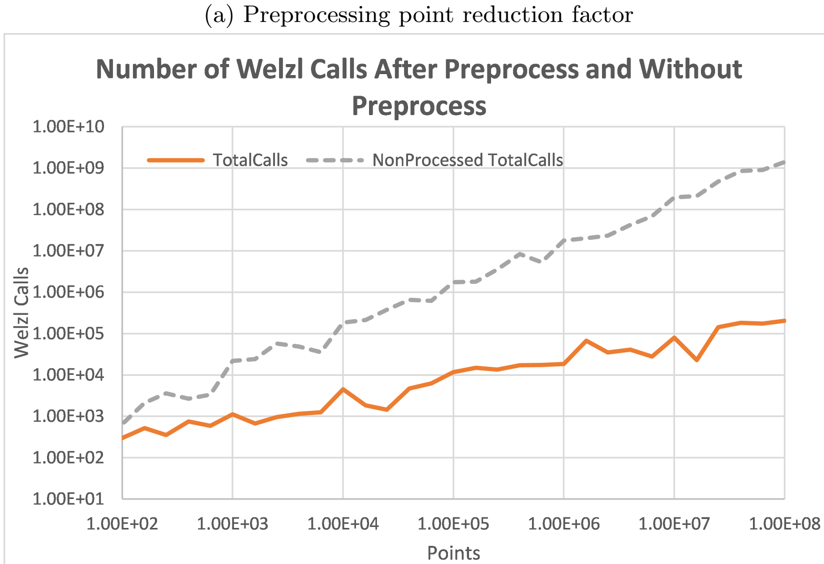


(a) Preprocessing point reduction factor



(b) Comparison of the recursive depth

Fig. 5: Reduction and recursive depth

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Appendix

This appendix presents related papers to the Smallest Enclosing Ball problem.

Agarwal[1], Cavaleiro[2][3], Cazals[4], Chen[5], Drager[6], Edelsbrunner[7], Efrat[8][9], Elzinga[10], wiki:[71][70][72], Fischer[12][13][11], Friedman[14], Gaertner[15], Gao[16], Goaoc[17], Har-Peled[18], Jiang[20][19], Kallberg[21], Karmakar[22][23], Krivosija[24], Larsson[25], Li[26], Liu[27], Martinetz[28], Martyn[29], Matousek[30], Megiddo[31], Mordukhovich[32], Mukherjee[33], Munteanu[34], Nam[35], Nielsen[38][39][40], Nielsen[41][42][37][36], Nock[43], Pan[44], Pronzato[45], Ritter[46], Saha[47], Shen[48], Shenmaier[49], Shi[50], Skyum[60], Smolik[61], Sylvester[62], Tao[63], Wang[64][65], Wei[66], Welzl[67][68][69], Xu[74][73], Yildirim[75], Zhou[76][77][78].

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