Advantages of interval modification of NURBS curves in modeling uncertain boundary shape in boundary value problems

Marta Kapturczak[0000−0003−0156−9904], Eugeniusz Zieniuk[0000−0002−−6395−5096], and Andrzej Kużelewski[0000−0003−2247−2714]

Institute of Computer Science, University of Bialystok, Białystok, Poland
{mkapturczak, ezieniuk, akuzel}@ii.uwb.edu.pl

Abstract. In this paper, the advantages of interval modification of NURBS curves for modeling uncertainly defined boundary shapes in boundary value problems, are presented. The different interval techniques for modeling the uncertainty of linear as well as curvilinear shapes are considered. The uncertainty of the boundary shape is defined using interval coordinates of control points. The knots and weights in the proposed interval modification of NURBS curves are defined exactly. Such a definition allows for modification of the uncertainly defined shape without any change of interval values. The interval NURBS curves are compared with other interval techniques. The correctness of modeling the shape uncertainty is confirmed by solving the problem using the interval parametric integral equations system method. Such solutions (obtained using a program implemented by authors) confirm the advantages of using interval NURBS curves for modeling the boundary shape uncertainty. The shape approximation is improved using less number of interval input data and the obtained solutions are correct and less over-estimated.

Keywords: NURBS curves · modeling uncertainty · interval arithmetic · boundary value problems · PIES

1 Introduction

All kinds of shapes can be modeled using computer graphics curves. Nowadays, the application of even a complex mathematical model to determine these curves, using computer techniques, is a very effective approach. This allows for more accurate and realistic modeling of any of the object shapes. Using analytical methods would be very troublesome and time-consuming.

The NURBS curves [1] are more and more frequently used in the boundary problems [2, 3]. These curves increase the accuracy of modeling the shape even with a small number of points. Additional parameters that increase the modeling possibilities are point weights and the knots vector. The weights determine the influence of the point on the curve and enable correct modeling of a circle or an ellipse. The knots allow to obtain corners and to change the degree of the curve.

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DOI: 10.1007/978-3-030-77980-1_33

The advantages of NURBS curves in modeling exactly defined problems [4, 5] motivated the authors to verify them in modeling uncertainly defined problems. In this paper, the boundary shape uncertainty in the boundary problems is modeled by NURBS curves using interval arithmetic [6, 7]. For this purpose, the control points' coordinates are defined using interval numbers. Consideration of the uncertainty (e.g. measurement errors) is a better approximation of reality.

The interval NURBS curves are compared with interval linear segments and interval Bézier curves to emphasize their advantages in modeling the boundary shape uncertainty. The impact of such modeling on the interval solutions of the problem is also analyzed. The mentioned modeling methods with the strategy of its inclusion into the mathematical formalism of the interval parametric integral equation system (interval PIES) [8] are presented below.

2 Modeling the boundary shape uncertainty

Direct application of classical or directed interval arithmetic [6, 7] in modeling boundary problems with any, uncertainly defined boundary shape is troublesome even with linear segments. A detailed description of the arising problems is presented in [9]. Among others, there is a lack of continuity between boundary segments (unrealistic problems are considered). Modeling the shape in different quadrants of the Cartesian coordinate system gives different results. Therefore, the authors proposed a modification of directed interval arithmetic by shifting arithmetic operators to the positive semi-axis as follows (for multiplication):

$$\boldsymbol{x} \cdot \boldsymbol{y} = \begin{cases} \boldsymbol{x}_s \cdot \boldsymbol{y}_s - \boldsymbol{x}_s \cdot y_m - x_m \cdot \boldsymbol{y}_s + x_m \cdot y_m & \text{for } \boldsymbol{x} \le 0, \boldsymbol{y} \le 0 \\ \boldsymbol{x}_s \cdot \boldsymbol{y} - x_m \cdot \boldsymbol{y} & \text{for } \boldsymbol{x} > 0, \boldsymbol{y} \le 0 \\ \boldsymbol{x} \cdot \boldsymbol{y}_s - \boldsymbol{x} \cdot y_m & \text{for } \boldsymbol{x} \le 0, \boldsymbol{y} > 0 \\ \boldsymbol{x} \cdot \boldsymbol{y} & \text{for } \boldsymbol{x} > 0, \boldsymbol{y} > 0 \end{cases}, \tag{1}$$

where $(\cdot)$ is an interval multiplication and for any $\boldsymbol{a} = [\underline{a}, \overline{a}]$ can be defined $\boldsymbol{a}_s = \boldsymbol{a} + a_m$ and $a_m = \begin{cases} |\overline{a}| & \text{for } \overline{a} > \underline{a} \\ |\underline{a}| & \text{for } \overline{a} < \underline{a} \end{cases}$, where $\begin{cases} \boldsymbol{a} > 0 \rightarrow \underline{a} > 0 \text{ and } \overline{a} > 0 \\ \boldsymbol{a} \le 0 \rightarrow \underline{a} < 0 \text{ or } \overline{a} < 0 \end{cases}$.

Significant advantages of exactly defined NURBS curves [1] in PIES are presented in [4, 5]. Therefore in this paper, for modeling uncertainly defined boundary shape, it is decided to verify the effectiveness of its interval modification:

$$\boldsymbol{S}_m(s) = \frac{\sum\limits_{i=0}^{n} w_i \boldsymbol{P}_i N_i^k(s)}{\sum\limits_{i=0}^{n} w_i N_i^k(s)} \quad \text{dla} \quad t_k \le s \le t_{n+1}, \tag{2}$$

where $\boldsymbol{P}_i (i = 0, 1, \ldots, n)$ are the interval control points, $w_i (i = 0, 1, \ldots, n)$ are exactly defined weights corresponding to points, and the base function $N_i^k(s)$ of k degree is exactly defined as normalized B-spline blending function [1]. Its definition requires also exactly defined elements of the knot vector.

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DOI: 10.1007/978-3-030-77980-1_33

The advantage of such an interval modification of NURBS curves is the possibility to change the uncertainly defined boundary shape using only exactly defined knots and weights (without changing the interval coordinates of control points). In Fig. 1 the examples of such kind of shape modifications are presented.

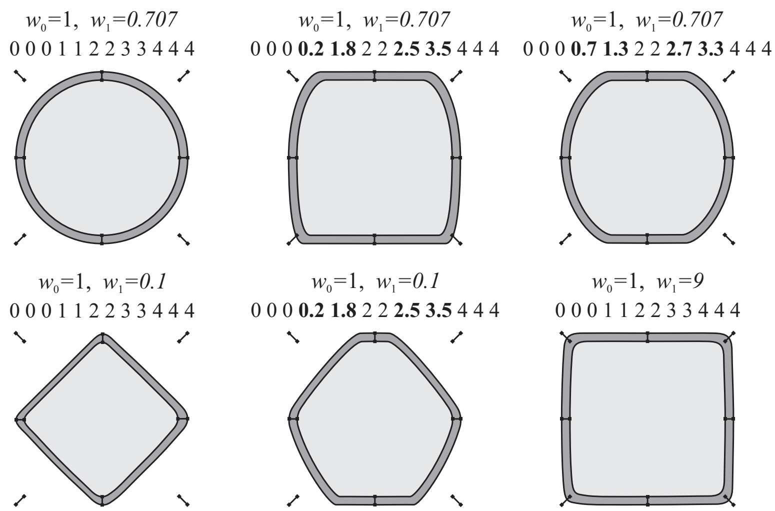

Fig. 1. Modification of interval shape using exactly defined weights and knots.

So, interval NURBS curves make modification much easier. Additionally, using the second-degree curve, a fewer amount of interval input data can be used what means fewer calculations on these numbers. This significantly reduces overestimation and improves obtained interval solutions.

3 Inclusion of interval curves into interval PIES method

The effectiveness of the PIES method and the accuracy of its solutions have been confirmed for exactly defined problems [10, 11]. Therefore, in this paper, to obtain solutions on the boundary (of uncertainly defined two-dimensional problem modeled by Laplace's equation), the interval PIES method [8] is proposed:

$$\frac{1}{2}u_l(s_1) = \sum_{j=1}^{n} \int\limits_{\widehat{s}_{j-1}}^{\widehat{s}_j} \left\{ \boldsymbol{U}^*_{lj}(s_1,s)p_j(s) - \boldsymbol{P}^*_{lj}(s_1,s)u_j(s) \right\} \boldsymbol{J}_j(s)ds, \tag{3}$$

where $\widehat{s}_{l-1} \leq s_1 \leq \widehat{s}_l$, $\widehat{s}_{j-1} \leq s \leq \widehat{s}_j$ are defined exactly in a parametric coordinate system and correspond to the beginning and the end of the segment of the interval curve $\boldsymbol{S}_m$ (where m = j, l).

The functions $p_j(s)$, $u_j(s)$ are parametric boundary functions on individual segments $\boldsymbol{S}_j$ of the interval boundary. One of these will be given as boundary

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conditions, while the other will be searched for as a result of the numerical solution of the interval PIES. In this paper, to analyze only the influence of the boundary shape uncertainty, the boundary conditions will be defined exactly.

To include the uncertainly defined boundary shape in PIES, the kernels should be modified. Hence, they will be defined as following interval functions $\boldsymbol{U}^*_{lj}(s_1,s)=[\underline{U}^*_{lj}(s_1,s),\overline{U}^*_{lj}(s_1,s)], \boldsymbol{P}^*_{lj}(s_1,s)=[\underline{P}^*_{lj}(s_1,s),\overline{P}^*_{lj}(s_1,s)]$:

$$\boldsymbol{U}^*_{lj}(s_1,s)=\frac{1}{2\pi}\ln\frac{1}{[\boldsymbol{\eta}_1^2+\boldsymbol{\eta}_2^2]^{0.5}}, \boldsymbol{P}^*_{lj}(s_1,s)=\frac{1}{2\pi}\frac{\boldsymbol{\eta}_1\boldsymbol{n}_1(s)+\boldsymbol{\eta}_2\boldsymbol{n}_2(s)}{\boldsymbol{\eta}_1^2+\boldsymbol{\eta}_2^2}, \tag{4}$$

where $\boldsymbol{n}_1(s)=[\underline{n}_1(s),\overline{n}_1(s)]$, $\boldsymbol{n}_2(s)=[\underline{n}_2(s),\overline{n}_2(s)]$ are the interval components of $\boldsymbol{n}(s)$ - the normal vector to the interval segment $\boldsymbol{S}_j$. The kernels analytically include the boundary shape uncertainty into its mathematical formalism. Such shape is defined as relation between interval segments $\boldsymbol{S}_m(m=l,j=1,2,...,n)$:

$$\boldsymbol{\eta}_1=\boldsymbol{S}_l^{(1)}(s_1)-\boldsymbol{S}_j^{(1)}(s_1), \boldsymbol{\eta}_2=\boldsymbol{S}_l^{(2)}(s_1)-\boldsymbol{S}_j^{(2)}(s_1). \tag{5}$$

The uncertainty of the boundary shape should be also included in the Jacobian $\boldsymbol{J}_j(s)=[\underline{J}_j(s),\overline{J}_j(s)]$ for the segment of the interval curve $\boldsymbol{S}_j(s)$.

The PIES numerical solution does not require classical discretization, unlike the boundary integral equation (BIE). To include the boundary uncertainty directly in functions (4) the interval segments will be defined by interval NURBS curves (2). The interval Bézier curves [12, 13] of the second and third-degree are used for comparison:

$$\boldsymbol{S}_m(s)=(1-s)^2\boldsymbol{P}_0+2(1-s)\boldsymbol{P}_1+s^2\boldsymbol{P}_2, \tag{6}$$

$$\boldsymbol{S}_m(s)=(1-s)^3\boldsymbol{P}_0+3(1-s)^2\boldsymbol{P}_1+(1-s)s^2\boldsymbol{P}_2+s^3\boldsymbol{P}_3, \tag{7}$$

where $m=l,j$. The second-degree curve (6) depends on three interval points: approximating ($\boldsymbol{P}_1$) and interpolating ($\boldsymbol{P}_0,\boldsymbol{P}_2$) and the third-degree curve (7) on four points respectively: approximating ($\boldsymbol{P}_1,\boldsymbol{P}_2$) and interpolating ($\boldsymbol{P}_0,\boldsymbol{P}_3$).

4 Comparison of interval PIES solutions

The shape of the first example is modeled using interval linear segments (Fig. 2a) and using a second-degree interval NURBS curve (Fig. 2b). The Dirichlet boundary conditions $u=0.5(x^2+y^2)$ are defined. The analytical solution [14] of the problem with error obtained by total differential method [15] is defined as:

$$u_a=\frac{x^3-3xy^2}{2a}+\frac{2a^2}{27}, \quad \Delta u_a=\left|\frac{3xy^2-x^3}{2a^2}+\frac{4a}{27}\right||\Delta a|, \tag{8}$$

where the height of the triangle is uncertainly defined as $\boldsymbol{a}=[\underline{a},\overline{a}]=[2.9,3.1]$, then $a=0.5(\overline{a}+\underline{a})$ and $\Delta a=0.5|\overline{a}-\underline{a}|$. The analytical interval solution will be defined as: $\boldsymbol{u}_a=[u_a-\Delta u_a,u_a+\Delta u_a]$. The interval PIES solutions with both modeling methods and the interval analytical solutions are presented in Tab. 1.

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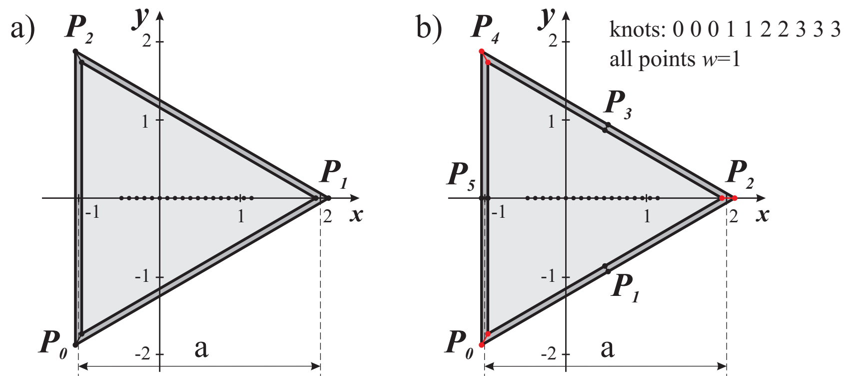

Fig. 2. Boundary shape obtained using interval a) linear segments, b) NURBS curves.

Table 1. Comparison of interval PIES solutions in domain (example from Fig. 2).

$y = 0$ x	**Analytical total differential**		**Interval PIES NURBS**		**Interval PIES linear**	
-0.4	0.611	0.701	0.612	0.702	0.612	0.702
-0.1	0.622	0.711	0.623	0.712	0.623	0.712
0.2	0.624	0.712	0.624	0.713	0.624	0.713
0.5	0.644	0.731	0.645	0.732	0.645	0.732
0.8	0.710	0.794	0.711	0.794	0.711	0.794
1.1	0.851	0.926	0.852	0.927	0.852	0.927

The interval PIES solutions with linear segments are almost equal to those with the interval NURBS curves (the average relative error is $3 \cdot 10^{-7}\%$). The average relative error of solutions in comparison to interval analytical ones is 0.1%. Obtained solutions are correct and almost without overestimation.

The correctness of the algorithm has been confirmed, so to emphasize the advantages of the strategy the problem with elliptical domain is also considered. The shape is modeled using the second-degree interval NURBS curve (Fig. 3a) with double knots (0 0 0 1 1 2 2 3 3 4 4 4) and using interval Bézier curves of second (Fig. 3 b) [13] and third degree (Fig. 3 b). The Dirichlet boundary conditions $u = 0.5(x^2 + y^2)$ are defined and exact analytical solution is [14]:

$$u_a = \frac{x^2 + y^2}{2} - \frac{a^2 b^2 (\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1)}{a^2 + b^2}, \tag{9}$$

where semi-major axis a and semi-minor axis b of the ellipse are defined as $\boldsymbol{a} = [\underline{a}, \overline{a}] = [1.95, 2.05]$ and $\boldsymbol{b} = [\underline{b}, \overline{b}] = [0.9, 1.1]$. Therefore, analogically to the previous example, interval analytical solutions (with error Δu_a obtained using the total differential method [15]) are presented as $\boldsymbol{u}_a = [u_a - \Delta u_a, u_a + \Delta u_a]$.

The average of the lower and upper bound relative error of interval PIES solutions in comparison with analytical ones are presented in Fig. 4. The solutions obtained using the second-degree NURBS curves (8 interval points) and

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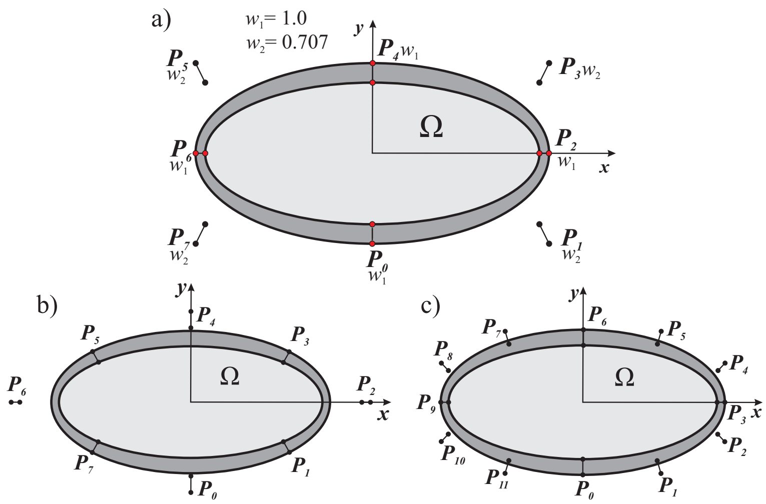

Fig. 3. Uncertainly defined elliptical shape of the boundary modeled by interval curves a) NURBS II degree, b) Bézier II degree, c) Bézier III degree.

third-degree Bézier curves (12 interval points) are almost equal (maximum error 0.8%). The maximum error after application of second-degree Bézier curves is about 1.8%. Therefore, the interval NURBS curves are not only easy to model and modify but also the obtained results are correct and less overestimated.

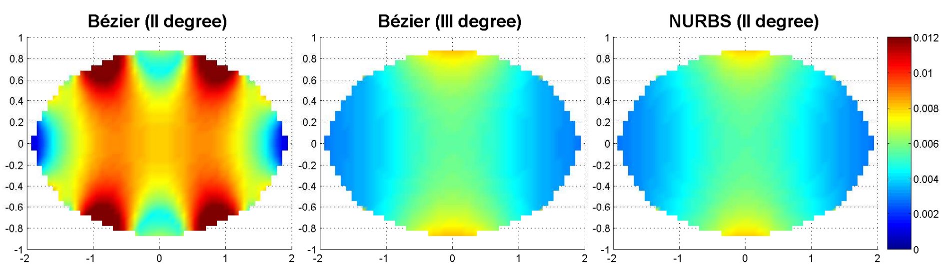

Fig. 4. Average relative error of interval PIES solutions in domain Fig. 3.

5 Conclusions

This paper presents the advantages of using interval modification of NURBS curves in modeling uncertainly defined boundary problems. Exactly defined weights and knots allow modifying the curve without changing the interval points. The modeling method is unified using one second-degree interval NURBS curve, without separate modeling of segments (using linear segments or Bézier curves). The advantages of the NURBS curves are emphasized by analyzing the

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modeling method's influence on the accuracy of the solutions (obtained using the interval PIES method). Interval NURBS curves are compared to linear segments and Bézier curves (second and third-degree). The application of interval NURBS curves gives correct solutions. Its definition requires a smaller amount of interval input data to obtain less overestimated interval solutions. So, the accuracy improvement of modeling the boundary shape uncertainty (using interval NURBS curves), improves the accuracy of the obtained interval solutions.

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