Low-dimensional Decompositions for Nonlinear Finite Impulse Response Modeling ⋆

Maciej Filiński[1 [0000−0001−9785−5133]], Paweł Wachel[1 [0000−0002−7353−2310]], and Koen Tiels[2 [0000−0001−9279−110X]]

[1] Wrocław University of Science and Technology, Wrocław 50-556, Poland
Maciej.Filinski@pwr.edu.pl, Pawel.Wachel@pwr.edu.pl
[2] Eindhoven University of Technology, 5600 MB Eindhoven, The Netherlands
K.Tiels@tue.nl

Abstract. This paper proposes a new decomposition technique for the general class of Non-linear Finite Impulse Response (NFIR) systems. Based on the estimates of projection operators, we construct a set of coefficients, sensitive to the separated internal system components with short-term memory, both linear and nonlinear. The proposed technique allows for the internal structure inference in the presence of unknown additive disturbance on the system output and for a class of arbitrary but bounded nonlinear characteristics.
The results of numerical experiments, shown and discussed in the paper, indicate applicability of the method for different types of nonlinear characteristics in the system.

Keywords: NFIR systems · short-term memory · structure detection

1 Introduction

Continuous development of technology results in new challenges in various engineering fields, but some basic fundamental problems remain invariably relevant and, to some extent, open. Among them, one can indicate modelling of nonlinear phenomena, [10], often encountered in various types of cyber-physical systems [9], technical processes, etc. In this paper, we focus on a modelling task for a wide class of stationary nonlinear dynamical systems with almost unknown internal structure. Assuming only that the system at hand has a finite memory and bounded non-linearity, we discuss a family of Non-linear Finite Impulse Response (NFIR) objects [5, 6, 11]. However, rather than direct identification, our goal is to reveal (if it exists) their hidden internal structure, *cf.* [4, 1], leading to the decomposition of the system into smaller, short-term memory linear or non-linear, elements. Such an approach leads to at least two benefits. Firstly, it potentially simplifies system identification through the reduction of problem dimensionality. Secondly, it supports inference about the role and properties of the particular short-term memory substructures. Our main motivation is that

⋆ Supported by Wrocław University of Science and Technology

ICCS Camera Ready Version 2021
To cite this paper please use the final published version:
DOI: 10.1007/978-3-030-77977-1_28

estimation of the separability of the system can be helpful in choosing a suitable model structure before estimating the system. The proposed method is based on the projection operators technique [7, 8] and allows to infer about systems with almost arbitrary non-linearity in the presence of additive output disturbance.

The paper is organized as follows. Section 2 formally introduces the class of NFIR systems and defines the general decomposition problem. Sections 3 and 4 introduce the proposed methodology, its motivation, and dedicated algorithms. Sections 5 and 6 present results of numerical experiments and final remarks.

2 Problem statement

We consider single-input single-output (SISO), time-invariant, Non-linear Finite Impulse Response (NFIR) systems with memory length d, described by

$$y_n = g(v_n) + e_n \tag{1}$$

$$v_n = [u_n, u_{n-1}, \ldots, u_{n-d}], \tag{2}$$

where $g : \mathbb{R}^{d+1} \to \mathbb{R}^1$ is an unknown nonlinear characteristic, u_n, y_n are the input and output signals, respectively, and e_n is an additive output disturbance; *cf.* Fig. 1. Regarding the system (1)–(2), input signal and output noise, we assume:

A1. The system has a finite memory of known length $d < \infty$, and unknown but bounded nonlinear characteristic $g : \mathbb{R}^{d+1} \to \mathbb{R}^1$.
A2. Input u_n is an *i.i.d.* sequence, uniformly distributed on the interval $[0, 1]$.
A3. The noise e_n is a zero mean *i.i.d.* sequence with a finite variance $\sigma_e^2 < \infty$ and is independent of u_n.

The requirements above are general in this sense that (for large enough d) the considered NFIR systems well approximate a general class of fading memory nonlinear objects, *cf.* [2]. We admit that assumption **A2** is rather restrictive and, in practice, allows to apply the proposed method if the system at hand can be actively excited by the user defined inputs, *cf.* [3]. Assumption **A3** is a standard assumption in literature.

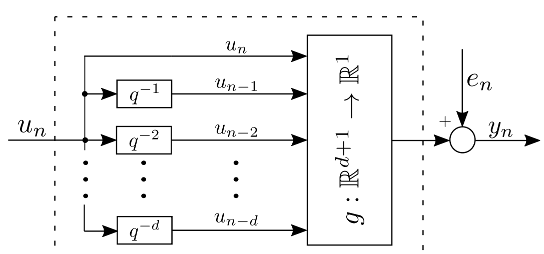

Fig. 1. The system under consideration. The non-linearity $g(\cdot)$ is shown as a mono-block, as its internal structure is unknown; q is the time-shift operator

ICCS Camera Ready Version 2021
To cite this paper please use the final published version:
DOI: 10.1007/978-3-030-77977-1_28

Based on the set of measurements of the input and output of the system, $\{(u_1, y_1), (u_2, y_2), \ldots, (u_N, y_N)\}$, captured in a steady state, the aim is to investigate a potential separability of characteristic $g(\cdot)$ for its alternative representation, composed of additive, Short-Term Memory Nonlinear blocks (STMN), *cf.* exemplary decompositions in Fig. 2b, 2c. Here, short-term memory refers to splitting nonlinear characteristic $g(\cdot)$, which has memory length d, into a sum of two or more nonlinear characteristics g_{ξ_i}, each having a memory length shorter than d. A more formal description is provided in the next sections.

3 Separation algorithm

We begin with a simple observation: due to assumptions **A1** and **A2**, $g(\cdot)$ is square integrable in a domain of system input. Let $\{1, 2, \ldots, d+1\}$ be a set denoting indices of the consecutive arguments of $g(\cdot)$, and $\xi_1 \subseteq \{1, 2, \ldots, d+1\}$ be its arbitrary subset. Let $\xi_2 = \{1, 2, \ldots, d+1\} \setminus \xi_1$ be a complement of ξ_1. We say that $g(\cdot)$ is separable with respect to (wrt) $\{\xi_1, \xi_2\}$ if $g \equiv g_{\xi_1} + g_{\xi_2}$, where g_{ξ_j} is a function depending *only* on the variables with indices from ξ_j. Recursive extension of this rule defines separability of $g(\cdot)$ for extended separation schemes $\{\xi_1, \xi_2, \ldots, \xi_\mu; \mu \leq d+1\}$. We note that $g(\cdot)$ in (1)–(2) is separable wrt $\{\xi_1, \xi_2, \ldots, \xi_\mu\}$ iff $S(\xi_1, \xi_2, \ldots, \xi_\mu) := S_1 - S_2$ equals zero [7, Th. 2] (*cf.* [8]), with

$$S_1 = \int_{[0,1]^{(d+1)}} \int_{[0,1]^{(d+1)}} g(\mathbf{s})(g(\mathbf{s}) + (\mu - 1)g(\mathbf{t}))d\mathbf{s}d\mathbf{t} \tag{3}$$

$$S_2 = \int_{[0,1]^{(d+1)}} \int_{[0,1]^{(d+1)}} g(\mathbf{s}) \sum_{j=1}^{\mu} g(\mathbf{s}_{\xi_j}, \mathbf{t}_{\{1,2,\ldots,d+1\}\setminus\xi_j})d\mathbf{s}d\mathbf{t}, \tag{4}$$

and with $g(\mathbf{s}_{\xi_j}, \mathbf{t}_{\{1,2,\ldots,d+1\}\setminus\xi_j})$ denoting the value of $g(\cdot)$ for the argument composed of ξ_j–indexed entries of $\mathbf{s}$ and $(\{1, 2, \ldots, d+1\} \setminus \xi_j)$–indexed entries of $\mathbf{t}$.

Clearly, the above integrals cannot be evaluated analytically without system knowledge. Yet, based on their stochastic interpretation as expectations wrt the uniform probability distribution, one can estimate S_1 using input-output observations of the system (*cf.* assumption **A2**). The corresponding estimator is

$$\hat{S}_1 = \frac{1}{|\mathbf{I}|} \sum_{i \in \mathbf{I}} y_i(y_i + (\mu - 1)y_{i+c}), \tag{5}$$

where $\mathbf{I} = \{n : n = (d+1), 2(d+1), 3(d+1), \ldots; d+1 \leq n \leq c(d+1)\}$, $c = \lfloor N/(2(d+1)) \rfloor$ and $|\mathbf{I}|$ is the cardinality of set $\mathbf{I}$.

Slightly more effort is needed to estimate integral S_2, since due to argument $(\mathbf{s}_{\xi_j}, \mathbf{t}_{\{1,2,\ldots,d+1\}\setminus\xi_j})$ of non-linearity g, the system has to be excited with a properly designed input, determined by the actually probed separation scheme ξ_j. We, therefore, design a supplementary input sequence U according to Algorithm 1, excite the system with U and collect the corresponding output, denoted as Y.

ICCS Camera Ready Version 2021
To cite this paper please use the final published version:
DOI: 10.1007/978-3-030-77977-1_28

Algorithm 1 Data Generation For Active Experiment

Input: $\{u_1, u_2, \ldots, u_N\}$, candidate separation scheme $\{\xi_1, \xi_2, \ldots, \xi_\mu\}$
$c := \lfloor N/2 \rfloor$
for $i = 1$ **to** μ **do**
  for $j = 1$ **to** c **do**
    for $k = 1$ **to** $d + 1$ **do**
      if $k \in \xi_i$ **then** $U_{(i-1)(d+1)c+(j-1)(d+1)+k} := u_{c(d+1)+k+(d+1)(j-1)}$
      else $U_{(i-1)(d+1)c+(j-1)(d+1)+d+1-(k-1)} := u_{d+1-(k-1)+(d+1)(j-1)}$
      end if
    end for
  end for
end for
Output: Supplementary input sequence U.

Based on the active experiment outcome, the following estimate of integral $\hat{S}_2$ can be defined

$$\hat{S}_2 = \frac{1}{|\mathbf{I}|} \sum_{i \in \mathbf{I}} \sum_{j=1}^{\mu} y_i Y_{c(j-1)(d+1)+i}. \tag{6}$$

Finally, as a resulting estimate of separability coefficient $S(\xi_1, \xi_2, \ldots, \xi_\mu)$ we take $\hat{S}(\xi_1, \xi_2, \ldots, \xi_\mu) = \hat{S}_1 - \hat{S}_2$.

Remark 1. Theoretical analysis of estimates $\hat{S}_1$, $\hat{S}_2$ is out of scope of the paper. Here, we only note that $\hat{S}_1$ is in fact a *biased* estimate of S_1 (with a bias equal to the variance of the output noise, σ_e^2). In effect, $\hat{S}$ is a *biased* estimate of S, with $\text{bias}\{\hat{S}\} = \sigma_e^2$. This is exploited in the numerical experiments in Section 5.

4 Short-Term Memory Separation Searching

We are now about to apply empirical coefficient $\hat{S}$ in decomposition of NFIR systems (with total memory length d) into a parallel connection of *Short-Term Memory Nonlinear* blocks (STMN); *cf.* Fig. 2b, 2c. Assuming its existence, such a representation is equivalent to the requirement that the genuine separation scheme, $\{\xi_1, \xi_2, \ldots, \xi_\mu; \mu \leq d + 1\}$, is an *ordered* set of indices. For instance, for $d = 2$, $g(\cdot)$ could be separable with respect to $\xi_1 = \{1, 2\}$, $\xi_2 = \{3\}$, but not with respect to $\xi_1 = \{1, 3\}$ and $\xi_2 = \{2\}$. Hence, the resulting representation of the system, if exists, is composed of the Short-Term Memory blocks. In general, observe that if $g(\cdot)$ is separable with respect to $\xi_p = \{1, 2, \ldots, p\}$ and $\xi_q = \{1, 2, \ldots, q\}$, some $p < q$, then it is also separable with respect to $\{1, 2, \ldots, p\}$ and $\{p+1, p+2, \ldots, q\}$. Therefore, the proposed separation procedure estimates all the coefficients $\hat{S}$ for $\xi_p = \{1, \ldots, p\}$, vs. its complement for $p = 1, 2, \ldots, d+1$, and the outcome is next used as a recommendation for the final inference about the considered NFIR system separation scheme.

ICCS Camera Ready Version 2021
To cite this paper please use the final published version:
DOI: 10.1007/978-3-030-77977-1_28

Algorithm 2 Short-Term Memory Separation Searching

1: **Input:** $\{(u_1, y_1), (u_2, y_2), \ldots, (u_N, y_N)\}$
2: **for** $p = 1$ **to** $d + 1$ **do**
3: Design supplementary input U: apply Algorithm 1. for $\xi_p = \{1, \ldots, p\}$.
4: Excite the system with U and measure the corresponding output sequence Y.
5: Compute $\hat{S} = \hat{S}_1 - \hat{S}_2$ according to (5)–(6).
6: **end for**
7: **Output:** $\hat{S}(\xi_1), \hat{S}(\xi_2), \ldots, \hat{S}(\xi_{d+1})$.

Remark 2. Note that $\hat{S}(\xi_{d+1})$ with $\xi_{d+1} = \{1, \ldots, d+1\}$ is an estimate of σ_e^2. Indeed, for the above-mentioned separation scheme, characteristic $g(\cdot)$ is 'separable', and therefore, due to biasness of $\hat{S}$ (see Remark 1), it converges to σ_e^2.

Clearly, any system (1)–(2) is 'separable' with respect to the full set of indices $\xi_{d+1} = \{1, 2, ..., d+1\}$, and therefore, based on the observation in Remark 2, we use $\hat{S}(\xi_{d+1})$ as a reference for the relative assessing of $\hat{S}(\xi_1), \hat{S}(\xi_2), \ldots, \hat{S}(\xi_d)$. If $g(\cdot)$ is not separable for some subset of indices, the corresponding value of $\hat{S}$ is high (for large enough N) with respect to $\hat{S}(\xi_{d+1})$. Hence, for the interpretational purposes, all the values $\hat{S}(\xi_1), \hat{S}(\xi_2), \ldots$ are scaled according to the formula $\bar{S}(\xi_p) = |\hat{S}(\xi_p)| / \sum_{i=1}^{d+1} |\hat{S}(\xi_i)|$. Finally, the comparison of $\bar{S}(\xi_{d+1})$ with respect to $\bar{S}(\xi_p)$ is used as the indicator of separability. Although the above approach is not yet theoretically justified, the results of numerical experiments are promising, as we show in the next section.

Remark 3. The proposed method has a simple construction and linear time complexity with respect to N, although the computing time strongly depends on the total system memory length d.

5 Numerical experiments

In this section, we present selected results of numerical experiments, performed for the NFIR systems with various types of admissible separation schemes and memory length $d = 9$. The following types of systems are considered: (A) non-separable system with non-linearity $g_A(v_n) = \sum_{i=1}^{d} (100(u_{n-i}^2 - u_{n-i+1})^2 + (u_{n-i} - 1)^2)$, see Fig.2a, (B) partially separated system with non-linearity $g_B(v_n) = \sum_{i=0}^{\lfloor (d+1)/2 \rfloor} (100(u_{n-2i+1} - u_{n-2i})^2)$, see Fig. 2b, and (C) fully-separable system $g_C(v_n) = \sum_{i=0}^{d} (u_{n-i}^2 - 10 \cos(2\pi u_{n-i}) + 10)$, see Fig. 2c. All the systems, A, B, C, are driven with uniformly distributed signal $\mathcal{U}[0, 1]$, scaled internally to $\mathcal{U}[-2, 2], \mathcal{U}[-3, 3], \mathcal{U}[-5.12, 5.12]$, respectively, *cf.* [7]. We ran the simulations for the three different numbers of samples N and three levels of output noise, as indicated in Table 1. According to the results (see Table 1), one can infer which sets of indices indicate separability. Note, that for $\bar{S}(\xi_{10})$ in the case of systems A and B the results are very small (but in fact $\bar{S}(\xi_{10}) > 0$). This is also the case for indices revealing separability. However, different type of the results are visible for system C, where most of the values are high compared to $\bar{S}(\xi_{10})$.

ICCS Camera Ready Version 2021
To cite this paper please use the final published version:
DOI: 10.1007/978-3-030-77977-1_28

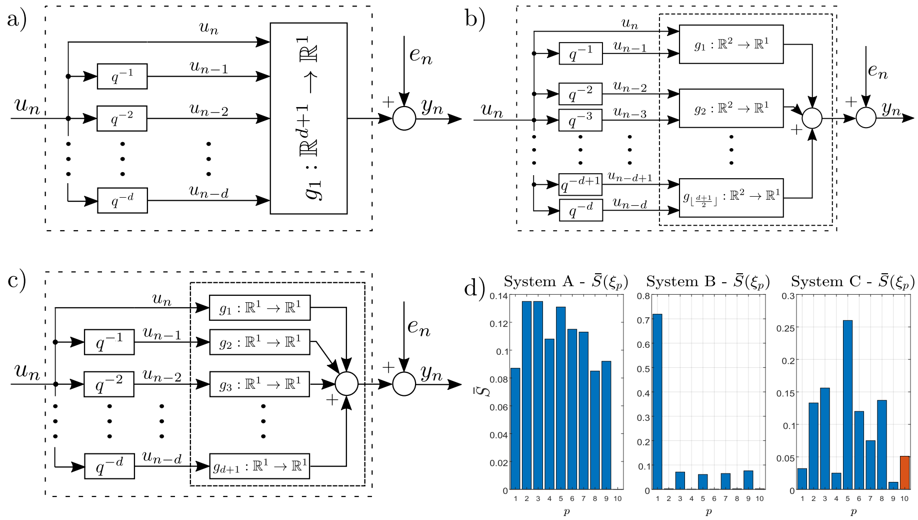

Fig. 2. Internal structure of a) a non-separable system A, b) a partially separable system B, and c) a fully separable system C. d) algorithm outcomes for the systems A, B, and C. Notice the small relative values of ξ_{10} for systems A, B and a higher one (with respect to $\xi_1, \ldots \xi_9$) for system C (in red).

6 Conclusions and future work

A new separation method was introduced for NFIR systems, representing a wide class of non-linear models with finite memory. The proposed method allows for the recovering of hidden *short-term memory structures* in the system under mild requirements regarding the form of non-linearity in the system. The method was investigated numerically for the systems with various levels of potential separability. According to the results shown in Table 1, the method correctly indicates separability patterns in the considered cases.

Future work includes a theoretical analysis in which separability is determined based on different threshold levels. Furthermore, the convergence rates of the proposed estimates, as well as the computational complexity of the method with respect to the system memory length will be thoroughly investigated.

References

1. Bai, E.W., Cerone, V., Regruto, D.: Separable inputs for the identification of block-oriented nonlinear systems. In: 2007 American Control Conference. pp. 1548–1553. IEEE (2007)
2. Boyd, S., Chua, L.: Fading memory and the problem of approximating nonlinear operators with Volterra series. IEEE Transactions on circuits and systems **32**(11), 1150–1161 (1985)
3. De Cock, A., Gevers, M., Schoukens, J.: D-optimal input design for nonlinear fir-type systems: A dispersion-based approach. Automatica **73**, 88–100 (2016)

DOI: 10.1007/978-3-030-77977-1_28

Table 1. The results of Algorithm 2 for non-separable system A, parially separable system B and fully-separable system C. The gray colored rows indicate true negative outcomes (lack of separability), whereas white rows represent true positive indications – possible separation of the system

		$N = 10^4$			$N = 10^5$			$N = 10^6$		
	σ_e^2	0	0.1	1	0	0.1	1	0	0.1	1
System A	ξ_1	0.028	0.009	0.003	0.068	0.145	0.154	0.099	0.068	0.087
	ξ_2	0.140	0.057	0.045	0.138	0.083	0.122	0.109	0.090	0.135
	ξ_3	0.227	0.039	0.204	0.108	0.035	0.079	0.130	0.136	0.135
	ξ_4	0.065	0.215	0.042	0.209	0.091	0.179	0.116	0.119	0.108
	ξ_5	0.139	0.205	0.066	0.039	0.122	0.100	0.096	0.133	0.131
	ξ_6	0.138	0.095	0.216	0.095	0.140	0.099	0.122	0.100	0.115
	ξ_7	0.160	0.161	0.249	0.099	0.182	0.080	0.086	0.108	0.113
	ξ_8	0.038	0.025	0.163	0.165	0.138	0.129	0.136	0.096	0.085
	ξ_9	0.065	0.195	0.011	0.080	0.064	0.058	0.107	0.150	0.092
	ξ_{10}	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
System B	ξ_1	0.869	0.773	0.681	0.736	0.676	0.688	0.732	0.728	0.719
	ξ_2	0.000	0.003	0.001	0.000	0.000	0.001	0.000	0.001	0.002
	ξ_3	0.018	0.010	0.039	0.058	0.113	0.077	0.069	0.071	0.071
	ξ_4	0.000	0.002	0.001	0.000	0.000	0.003	0.000	0.000	0.001
	ξ_5	0.024	0.097	0.094	0.060	0.082	0.076	0.065	0.063	0.061
	ξ_6	0.000	0.002	0.001	0.000	0.000	0.006	0.000	0.000	0.001
	ξ_7	0.052	0.050	0.058	0.066	0.052	0.055	0.070	0.069	0.065
	ξ_8	0.000	0.001	0.001	0.000	0.000	0.000	0.000	0.000	0.002
	ξ_9	0.038	0.062	0.114	0.081	0.076	0.093	0.064	0.068	0.076
	ξ_{10}	0.000	0.002	0.011	0.000	0.000	0.000	0.000	0.000	0.002
System C	ξ_1	0.125	0.105	0.036	0.100	0.069	0.192	0.111	0.002	0.032
	ξ_2	0.000	0.058	0.208	0.100	0.229	0.086	0.111	0.073	0.133
	ξ_3	0.125	0.012	0.103	0.100	0.068	0.024	0.111	0.161	0.156
	ξ_4	0.125	0.323	0.070	0.100	0.011	0.128	0.111	0.000	0.025
	ξ_5	0.125	0.134	0.045	0.100	0.028	0.129	0.000	0.236	0.260
	ξ_6	0.000	0.015	0.057	0.100	0.119	0.108	0.000	0.007	0.120
	ξ_7	0.125	0.073	0.024	0.100	0.068	0.055	0.111	0.232	0.075
	ξ_8	0.125	0.137	0.065	0.100	0.087	0.037	0.111	0.110	0.137
	ξ_9	0.125	0.017	0.236	0.100	0.212	0.095	0.222	0.023	0.011
	ξ_{10}	0.125	0.127	0.156	0.100	0.110	0.147	0.111	0.155	0.051

4. Decuyper, J., Tiels, K., Runacres, M.C., Schoukens, J.: Retrieving highly structured models starting from black-box nonlinear state-space models using polynomial decoupling. Mechanical Systems and Signal Processing **146**, 106966 (2019)
5. Enqvist, M., Ljung, L.: Linear models of nonlinear FIR systems with gaussian inputs. IFAC Proceedings Volumes **36**(16), 1873–1878 (2003)
6. Enqvist, M., Ljung, L.: Linear approximations of nonlinear FIR systems for separable input processes. Automatica **41**(3), 459–473 (2005)
7. Goda, T.: On the separability of multivariate functions. Mathematics and Computers in Simulation **159**, 210–219 (2019)
8. Kuo, F., Sloan, I., Wasilkowski, G., Woźniakowski, H.: On decompositions of multivariate functions. Mathematics of computation **79**(270), 953–966 (2010)
9. Lee, E.A.: Cyber physical systems: Design challenges. In: 2008 11th IEEE International Symposium on Object and Component-Oriented Real-Time Distributed Computing (ISORC). pp. 363–369. IEEE (2008)
10. Schoukens, J., Ljung, L.: Nonlinear system identification: A user-oriented road map. IEEE Control Systems Magazine **39**(6), 28–99 (2019)
11. Śliwiński, P., Marconato, A., Wachel, P., Birpoutsoukis, G.: Non-linear system modelling based on constrained Volterra series estimates. IET Control Theory & Applications **11**(15), 2623–2629 (2017)

ICCS Camera Ready Version 2021
To cite this paper please use the final published version:
DOI: 10.1007/978-3-030-77977-1_28