How to reach consensus? Better disagree with your neighbor. *

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Abstract. We study the basic first passage properties of a discrete one-dimensional mathematical model of opinion dynamics. The model studied here is a generalization of the original Sznajd model, which consists of introducing a new parameter p, being the probability of disagreement with the nearest neighbor in case of uncertainty. We study the model via Monte Carlo simulations and show that the exit probability does not change with the size of the system N, whereas the average exit time τ scales with N as $\tau \sim N^{\alpha}$. Moreover, we show that generally the consensus is reached more rapidly if agents disagree more often with their nearest neighbors in case of uncertainty.

Keywords: Mathematical Sociology · Complex Social Systems · Agent-Based Simulation · Opinion Dynamics.

1 Introduction

Agent based models (ABMs) are known as a tool which builds a bridge between micro and macro scale. Moreover, the macro outcome of the microscopic rules can be surprising and sometimes contra-intuitive, which is one of the key features of the complex systems, known as emergence [1].

In this short paper, we discuss one of such a non-intuitive result that is obtained within a simple opinion dynamics model, recently introduced in the review paper on the Sznajd model (SM) [12]. The model we study here can be treated as a generalization of the original one-dimensional SM and reduces in limiting cases to two versions of SM, which were usually studied in the literature [12].

We focus on one of the most studied issues in a field of opinion dynamics, namely reaching the consensus [11, 5, 6, 2, 4]. Our intuition says that the more you agree with your neighbors the easier consensus should be reached. Here, we will show that this is not necessarily true.

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2 The model

We study a system of N agents placed in the cells of the one-dimensional lattice with periodic boundary conditions. Each agent can be in one of two states, $S_i(t) = \pm 1$, which represent alternative opinions (yes/no, agree/disagree, etc.), that changes in time t due to interactions between agents. Each cell is occupied by an exactly one agent and agents cannot move. Therefore, equivalently we could describe the system as the one-dimensional grid of binary cells.

At each elementary update, a pair of neighboring cells $(i, i+1)$ is chosen at random and it influences two neighboring cells: one on the left side of a pair, i.e. $i-1$, and one on the right side, i.e. $i+2$. If pair $(i, i+1)$ is unanimous, i.e. $S_i(t) = S_{i+1}(t)$, then two neighbors take the same state as a pair: $S_{i-1}(t+\Delta t) = S_i(t)$, $S_{i+2}(t+\Delta t) = S_i(t)$, where Δt is a time period needed for a single update. Otherwise, if $S_i(t) = -S_{i+1}(t)$, cells $(i-1, i+2)$ take the opposite states to their nearest neighbors with probability p: $S_{i-1}(t+\Delta t) = -S_i(t)$, $S_{i+2}(t+\Delta t) = -S_{i+1}(t)$. It means that in case of uncertainty, agents disagree with their nearest neighbors with probability p: $p = 1$ corresponds to the original "United we stand, divided we fall" rule, whereas $p = 0$ to the rule, which has been mostly used in the literature within the Sznajd model [12]. As usual in this type of models, a time unit consists of N elementary updates, i.e. $N\Delta t = 1$.

We realize that the assumptions we made here, including one-dimensional structure and the lack of movement, describe only a limited number of real-life scenarios, such as opinion formation during the round-table discussions. In many other cases, such as discussion in various social media, the assumption of the lack of the movement would be still valid. However, the structure of the social network in such a case should be modeled by some heterogeneous, even multiplex graphs [11]. To model face-to-face contacts, allowing for agents' movement would be also an interesting idea but this would require an additional parameter describing the density of occupied nodes.

We study the model within Monte Carlo simulations from different initial conditions, parametrized by the concentration $c_0 \equiv c(0)$ of agents with positive opinion at time $t = 0$:

$$c(t) = \frac{N_+(t)}{N} = \frac{1}{2N} \sum_{i=1}^{N} (S_i(t) + 1), \tag{1}$$

where $N_+(t)$ is the number of agents with opinion $+1$ at time t. To define precisely initial conditions, besides the concentration of positive argents c_0, we have to decide on their spacial distribution. We consider two limiting cases, as in [12]:

- **Random**: choose randomly $N_+(0)$ out of N cells for agents with positive opinions. For large systems it is almost identical with a much simpler rule: for $i = 1$ to N, with probability $c_0 = N_+(0)/N$, set $S_i(0) = 1$ and, with complementary probability $1 - c_0$, set $S_i(0) = -1$.
- **Sorted**: for $i = 1$ to $N_+(0)$ set $S_i(0) = 1$ and for $i = N_+(0)$ to N set $S_i(0) = -1$.

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We average results over $L = 10^3$ independent samples, which means that for each set of parameters (p, c_0, N) we run L independent simulations. We stop simulation once the absorbing state is reached and then, we collect data:

- The final (absorbing) configuration, which for this model is one of the following [12]: (1) positive consensus $(+++++\cdots)$, (2) negative consensus $(-----\cdots)$ or (3) a stalemate state $(+-+-+\cdots)$. This allows us to calculate the probability of each absorbing state: P_+, P_-, P_{+-}, so called **exit probability** [7, 11].
- The time needed to reach the absorbing state, so called **exit time**, which allows us to calculate the average exit time τ [7].

3 Results

It was already shown in [12] that only for $p = 1$ the consensus or stalemate state can be reached, whereas for $p < 1$ only consensus is possible. For all values of p both the consensus, as well as the stalemate state are the fixed points, i.e. once the system is in one of these states it will never leave them. However, for $p < 1$ the stalemate state is unstable, so it can never be reached from other states.

In [12] the exit probability was measured within the Monte Carlo simulations for $N = 100$. Here, we checked it more systematically for different $N \geq 100$. It occurs that for any $p < 1$ the exit probability of the positive consensus can be approximated by:

- for random initial conditions (see Fig. 1):

$$P_+ = \frac{c_0^2}{c_0^2 + (1 - c_0)^2}, \tag{2}$$

- for sorted initial conditions $P_+ = c_0$.

Exactly the same results have been obtained previously for the original Sznajd model without disagreement rule, which corresponds to $p = 0$ [3, 9].

The second important first passage property, namely the exit time, was never measured so far for such a generalized model, but only for $p = 0$ [9]. It is obvious that $\tau = 0$ for $c_0 = 0$ or $c_0 = 1$ because the initial state is already an absorbing one. Moreover, we expect that τ has the maximum value for $c_0 = 0.5$ [8, 9]. Finally, we expect that the exit time for the random initial conditions should be a bit shorter than for the ordered ones [9].

What we do not know, is the role of the parameter p. As discussed above, the exit probability does not depend on p, as long as $p \neq 1$, for which additional absorbing stalemate state appears. It could indicate that p is an unnecessary parameter, which could be omitted. However, it turns out that it affects the exit time in a non-trivial way, as shown in Fig. 2.

For each c_0 there is an optimal value $p = p^* = p^*(c_0)$, for which consensus is the most rapid, i.e., the exit time is the shortest. This optimal value p^* is surprisingly high, as shown in Fig. 2. For example, if initially the number of

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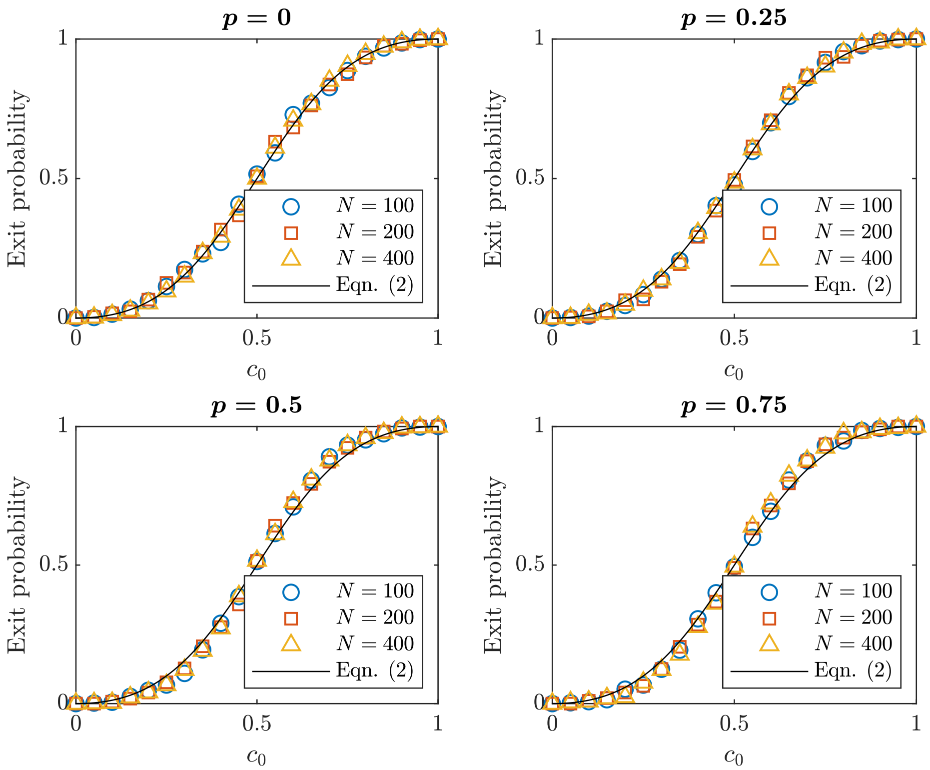

Fig. 1. Exit probability from random initial conditions for several system sizes N and four different values of p.

positive and negative opinions is more or less the same, $c_0 \approx 0.5$ the consensus is reached the most rapidly for $p = p^* \approx 0.9$. It means that consensus is reached faster if agents disagree with their nearest neighbors more often.

On contrary to the exit probability, the average exit time depends on the system size, what is rather expected, as shown in Fig. 3. For $p = 0$ it scales with an exponent 2, i.e. $\tau \sim L^2$, what has been already shown in [9]. For other values of p the scaling exponent $\alpha \approx 2$, but it is not exactly equal to 2.

4 Discussion

It is often claimed that the key lesson from agent-based modeling can be summarized by the sentence verbalized by Epstein [1, 10]: “We get macro-surprises despite complete micro-level knowledge”. In this paper, we show one of such surprises. We expected that for $p = 0$ the time evolution would be long, because then the change of the state is possible only if the source pair is unanimous. However, we did not expect that for $p > 0$ the exit time will decrease with

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an increasing p, and definitely not that the optimal value for the most rapid consensus will be so high.

The second surprise came with the finite size scaling. We expected the exit time to scale as the power law with an exponent $\alpha = 2$, but it occurs that it scales with $\alpha = 2$ only for the limiting values of $p = 0$ and $p = 1$. This result is far from being obvious and definitely needs deeper insight, what is a planned task for the future.

We are aware that Epstein's notion on macro-surprises can be criticized by saying that the level of surprise depends on the perceptiveness and experience of the researcher, the ability to find cause-effect relationships, etc. We cannot argue with that, just as we cannot state whether our intuition about what facilitates reaching consensus is the same as the intuition of the reader of this work. However, we hope that our short paper will inspire at least some readers to explore the model deeper, for example on more realistic social structure.

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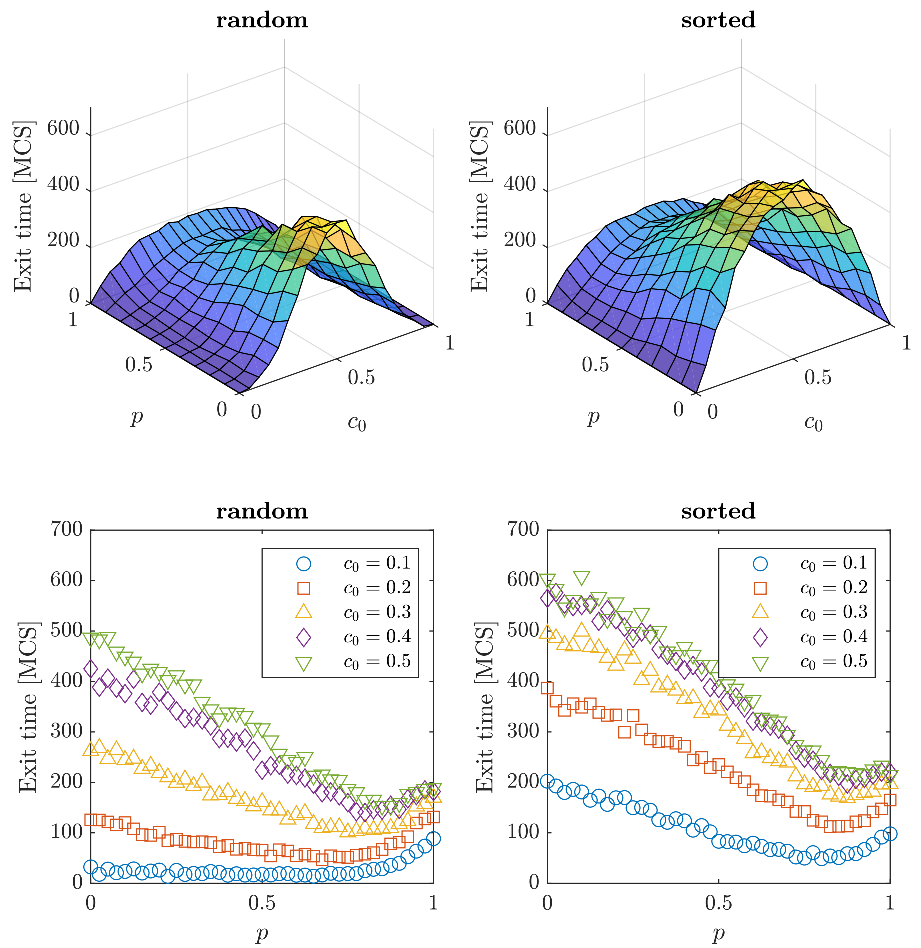

Fig. 2. Dependence between the average exit time to reach consensus and two model's parameters, for the system of size $N = 100$: initial concentration of positive opinions c_0 and the probability of disagreement p. Results from two types of initial conditions are shown: random (left panels) and sorted (right panels).

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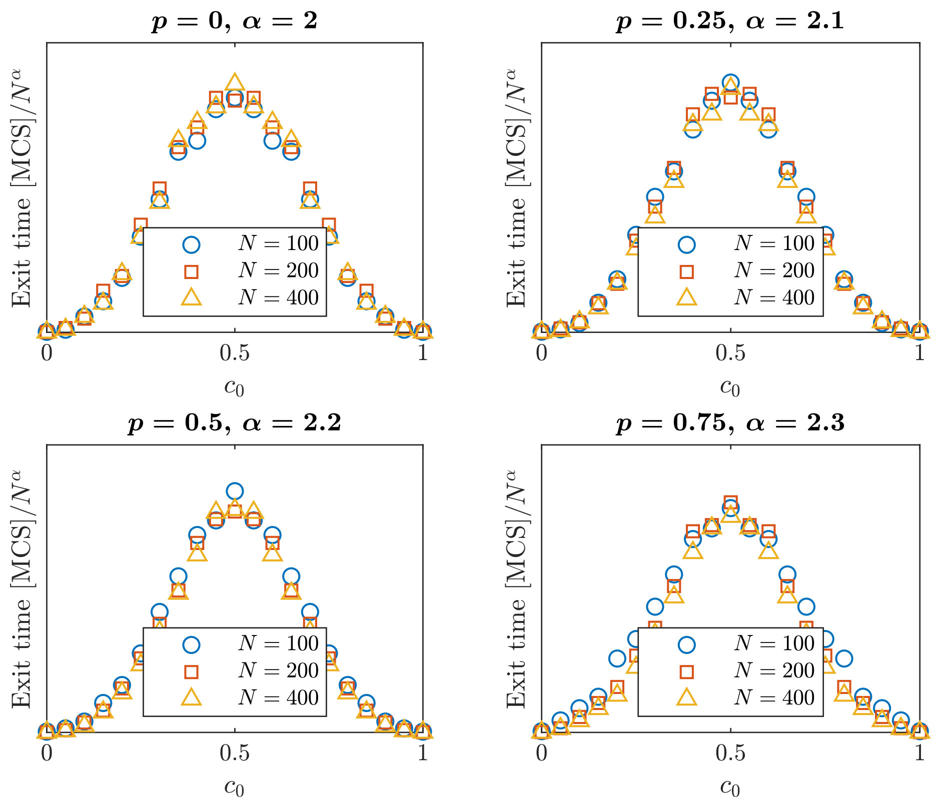

Fig. 3. Rescaled exit time from random initial conditions for several system sizes N and four different values of p.

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