Three-state opinion q-voter model with bounded confidence*

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Abstract. We study the q-voter model with bounded confidence on the complete graph. Agents can be in one of three states. Two types of agents behaviour are investigated: conformity and independence. We analyze whether this system is qualitatively different from a corresponding model without bounded confidence. The key result of this paper is that the system has two phase transitions: one between order-order phases and another between order-disorder phases.

Keywords: q-voter · complex systems · Monte Carlo simulations

1 Introduction

The q-voter model is widely used in the area of opinion dynamics [1–3]. Within the q-voter model opinions are usually binary dynamical variables. Only recently, a new version of the model with multi-state opinions was introduced [4, 5]. In [5] agents can change opinions without any limitations: all opinions are equivalent. The situation when opinions are not equivalent was analyzed by Stauffer for the Sznajd model [6]: one agent can convince another to its opinion only if they share similar opinions, i.e. not too distant from each other (this rule is known as a bounded confidence [7]). The simplest multi-state model with bounded confidence is a model with three opinions. In this case two opinions are considered as extreme and agents do not change their opinion from one extreme to another due to the bounded confidence in a single update. Agents with the middle opinion can change it to any other. One can think about this simple realisation of multi-state opinion model with bounded confidence in terms of political parties: left- and right-wing extreme parties and centrist party. There is empirical evidence that agents opinion has multidimensional nature [8] and cannot be reduced to simple *yes-no* case. The model with three-state opinion and bounded confidence is a step to make it more realistic. In the future even more states can be added.

In this paper we analyze to what extent the q-voter model with three-state opinion and bounded confidence is different from the one without bounded confidence.

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2 Methodology

Considered system consists of N agents. Each agent is characterized by the dynamical variable named opinion $o_i(t)$, $i = 1, ..., N$, where t denotes simulation time measured in Monte Carlo steps (MCS). Opinion takes one of three possible values: $o_i(t) = k \in \{1, 2, 3\}$ and only the following transitions between states are allowed:

$$1 \rightleftarrows 2 \rightleftarrows 3. \tag{1}$$

Transitions between states 1 and 3 are forbidden, so opinions 1 and 3 have only opinion 2 as a neighbouring one, whereas for opinion 2 both 1 and 3 are the neighbouring states.

Number of agents with opinion k at time t is denoted by $N_k(t)$, and their concentration is $c_k(t) = N_k(t)/N$. Agents are placed in the vertices of a complete graph (CG). We distinguish two types of behaviour: independence and conformity [2, 5]. Agents behave independently with probability p and conform to others with probability $1 - p$. For the latter case agent is influenced by the q-panel of unique neighbours (chosen without repetition). Changes of opinion are limited by restrictions from Eq.(1). In simulations we use random sequential updating, which means that in a single update only one agent can change its opinion. Pseudocod of a single update is presented below.

Algorithm 1: pseudocod

```
i ← random_int (1, N) ;
if p > random(0, 1) then
    i.state ← state (random_int (i.state − 1, i.state + 1));
else
    Find q neighbours;
    if states of neighbours are equal then
        if neighbour.state ∈ (state (i.state − 1) , state (i.state + 1))
         then
            i.state ← neighbour.state;
```

3 Mean-field approach

We investigate presented model on the CG - mainly because it allows for an exact theoretical calculations as CG is equivalent with the mean-field approach. Such an approach enables verification of the model by comparison of the Monte Carlo (MC) results with analytical ones. The dynamics of the system can be in general described as the flow of agents from one opinion to another. Opinions 1 and 3 have different dynamics than opinion 2 because of different number of neighbours (see Eq.(1)).

We want to calculate the flow between opinions for certain values of parameters and define the stationary state for each of those. Let us define concentration

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of a given state in the form of:

$$c_k(t) = \frac{N_k(t)}{N} = \frac{1}{N}\sum_{i=1}^{N} \delta\left(o_i(t), k\right), \tag{2}$$

where $\delta(i, j)$ is the Kronecker delta function.

In a random sequential updating the elementary change of concentration $c_k(t)$ for all k in a single update is $\Delta c = 1/N$. Concentration $c_k(t)$ can increase or decrease with corresponding probabilities

$$\gamma_k^+ = P(c_k \to c_k + \Delta c), \quad \gamma_k^- = P(c_k \to c_k - \Delta c). \tag{3}$$

Up to this moment we have been dealing with random variable $c_k(t)$. We can also write the evolution equation of the corresponding expected values. For $N \to \infty$ we assume that random variable $c_k(t)$ localizes to the expected value. The time evolution of the expected value of c_k is

$$c_k(t + \Delta t) = c_k(t) + \frac{1}{N}(\gamma_k^+ - \gamma_k^-). \tag{4}$$

Since there is N agents and one MCS means N individual updates then $N\Delta t = 1$ and $\Delta t = \frac{1}{N}$. If the system is large enough and $N \to \infty$ then Eq.(4) simplifies to

$$\frac{\partial c_k}{\partial t} = \gamma_k^+ - \gamma_k^-. \tag{5}$$

Now we write explicitly $\gamma_k^\pm$. With probability $(1-p)$ agent is a conformist. There are $N_{k'}$ agents with opinions in states different than k, but achievable for agent in state k. We randomly choose q neighbours and they all need to share opinion k. For the first neighbour there are N_k available agents out of $N - 1$. For every next neighbour there is one less unique agent. Finally the conformism part γ_{con}^+ and the outflow γ_{con}^- can be written as

$$\gamma_{k,con}^+ = \frac{N_{k'}}{N}(1-p)\left(\prod_{i=0}^{q} \frac{N_k - i}{N - 1 - i}\right), \gamma_{k,con}^- = \frac{N_k}{N}(1-p)\left(\prod_{i=0}^{q} \frac{N_{k'} - i}{N - 1 - i}\right). \tag{6}$$

With probability p agent is independent. There are $N_{k'}$ agents in different state than k and achievable for agent in state k. Due to the bounded confidence the independence term is different for $k = 1, 3$ and $k = 2$. Random choice of state for $k = 1, 3$ means that with probability $\frac{1}{2}$ we can change to $k' = 2$ or stay in the same state. For state $k = 2$ there are three options: $k' = 1, 3$ and preserving opinion, so each term has probability $\frac{1}{3}$. So for example $\gamma_{1,ind}$ yields

$$\gamma_{1,ind}^+ = \frac{N_2}{N}\frac{p}{3}, \quad \gamma_{1,ind}^- = \frac{N_1}{N}\frac{p}{2}. \tag{7}$$

Combining Eqs. (6-7) for state $k = 1$ we obtain

$$\gamma_1^+ = \frac{N_2}{N}\left((1-p)\left(\prod_{i=0}^{q} \frac{N_1 - i}{N - 1 - i}\right) + \frac{p}{3}\right), \tag{8}$$

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and

$$\gamma_1^- = \frac{N_1}{N}\left((1-p)\left(\prod_{i=0}^{q}\frac{N_2-i}{N-1-i}\right)+\frac{p}{2}\right). \tag{9}$$

When $N \to \infty$, the above equations simplify to

$$\gamma_1^+ = c_2\left((1-p)c_1^q+\frac{p}{3}\right), \quad \gamma_1^- = c_1\left((1-p)c_2^q+\frac{p}{2}\right). \tag{10}$$

Analogic formulas were derived for $k = 2, 3$. The time evolution of the system can be described via three equations:

$$\frac{\partial c_1}{\partial t} = c_2\left((1-p)\left(c_1^q\right)+\frac{p}{3}\right) - c_1\left((1-p)\left(c_2^q\right)+\frac{p}{2}\right), \tag{11}$$

$$\frac{\partial c_2}{\partial t} = (c_1+c_3)\left((1-p)\left(c_2^q\right)+\frac{p}{2}\right) - c_2\left((1-p)\left(c_1^q+c_3^q\right)+\frac{2p}{3}\right), \tag{12}$$

$$\frac{\partial c_3}{\partial t} = c_2\left((1-p)\left(c_3^q\right)+\frac{p}{3}\right) - c_3\left((1-p)\left(c_2^q\right)+\frac{p}{2}\right). \tag{13}$$

In the next Section we compare results obtained from MC simulations with numerical results from Eqs. (11-13).

4 Simulations

For the simulations we use the system of size $N = 25\,000$ and simulation time $t = 5\,000$ MCS. Results were averaged over 64 independent realisations.

Fig. 1 shows the plot of concentration c_1 against p for different values of q (see legend). Solid lines represent theoretical solutions for $c_1(p)$, whereas symbols denote the outcome of MC simulations. Coloured and empty symbols stand for different initial conditions: the former for $c_1(0) = 1$ and $c_2(0) = c_3(0) = 0$, the latter for $c_2(0) = 1$ and $c_1(0) = c_3(0) = 0$. It can be seen in more detail in the inset of Fig. 1, that those data sets are different but they tend to overlap for certain values of p. At first we focus on $c_1(p)$ for $q = 2$. Data maintains order up to $p \approx 0.19$ when c_1 drops from ~ 0.7 to ~ 0.1 suggesting some kind of phase transition in the system. For higher values of p, c_1 slowly and smoothly grows, through the inflection point to equilibrium value ~ 0.27. Final value of c_1 is not equal to $\frac{1}{3}$ because, as mentioned, opinions $k = 1, 3$ have only one neighbour while central opinion $k = 2$ has two neighbours. Data for $c_1(0) = 0$ (empty squares) slowly growths up to $p \approx 0.2$. For higher p both data sets overlap.

The data $c_1(p)$ for $q = 3$ (blue and empty circles) has very similar character to $q = 2$. The drop in c_1 value takes place earlier: for $p \approx 0.17$, after which coloured and empty circles overlap. Later growth is more rapid than for $q = 2$.

This character repeats for $q = 4$ (green and empty triangles). The only noticeable difference is that growth after $p \approx 0.13$ is much faster and an eye-inspection reveals the position of inflection point.

The data for $q = 5$ (magenta and empty diamonds) displays qualitatively different character. After the initial drop of c_1 for $p \approx 0.1$ we notice short, rapid

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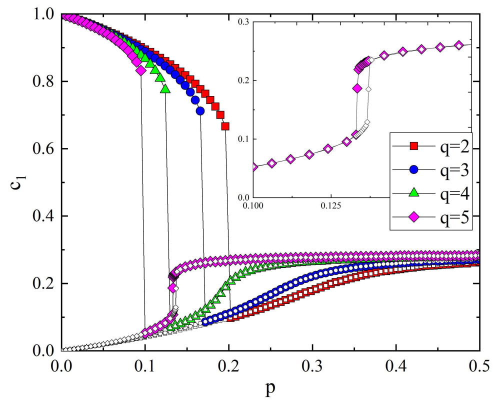

Fig. 1. Dependence between the stationary concentration c_1 of opinion 1 and probability of independence p for several values of q obtained from MC simulations (symbols) and numerical solutions of Eqs.(11-13) (solid lines). Coloured symbols stand for the initial configuration $c_1(0) = 1$. Empty symbols stand for $c_1(0) = 0$ (inset shows this in detail). Inset: hysteresis loop for $q = 5$, scaled up.

growth and another jump in value for $p \approx 0.14$. The second jump is discontinuous and displays hysteresis loop that can be seen via discrepancy between the two data sets: coloured and empty symbols. This part of the graph can be seen in more detail in the inset. This is the first case when we see double discontinuous transition: from one order to another (dominant $k = 1$ into dominant $k = 2$ or 3) and from order into disorder (dominant $k = 2$ or 3 into disorder).

The next key issue is what happens with the system after the first and the

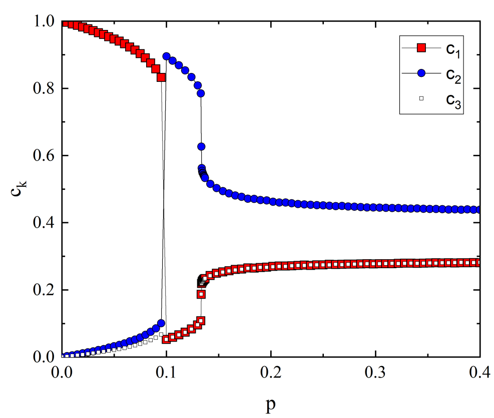

Fig. 2. Dependence between the stationary values of all opinions and the probability of independence p for $q = 5$.

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second transition. To explain this we plot the concentration of all states $c_{1,2,3}(p)$ for $q = 5$ - see Fig. 2. Initial condition was $c_1(0) = 1$ and $c_2(0) = c_3(0) = 0$. Up to $p = 0.1$ the concentration of states is the following: dominating state is $k = 1$, while states $k = 2, 3$ are in minority. Afterwards there is a rapid drop of c_1 and a rapid growth of c_2. For rather narrow region $p \in [0.1, 0.14]$ state $k = 2$ dominates while states $k = 1, 3$ are equal. Then there is a second rapid drop of value: this time c_2 drops while $c_{1,3}$ grow. For further increase of p there is a smooth, rather slight change of values of all concentrations reaching final value around $p \approx 0.3$.

Fig. 2 clearly shows that the first transition for $p \approx 0.1$ means the change of domination from state $k = 1$ into $k = 2$. The second transition for $p \approx 0.14$ means that state $k = 2$ looses decisive domination and for $p > p^* \approx 0.14$ almost disordered phase is reached with only slight dominance of the central opinion c_2 over the extremes $c_1 = c_3$. This results is in agreement with theoretically calculated values of stationary concentrations c_k for high p, namely $c_{1,3,st} = \frac{2}{7}$ and $c_{2,st} = \frac{3}{7}$.

To gain deeper insight into the behaviour of the system we present the

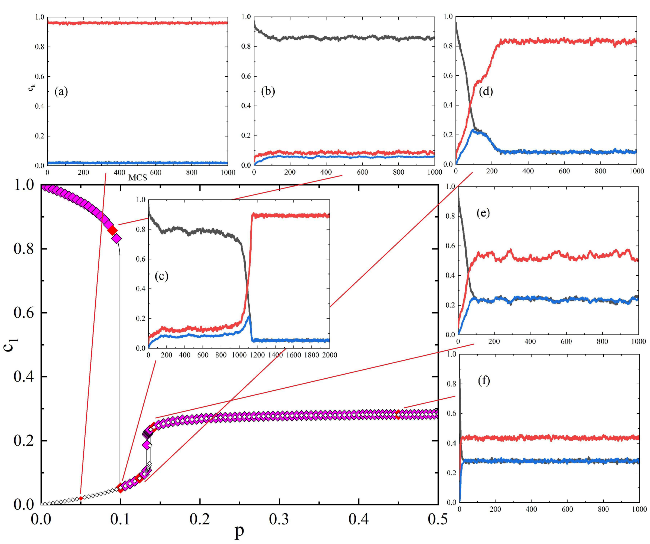

Fig. 3. Trajectories of $c_k(t)$ against MC time for $q = 5$ and various p. Each trajectory graph is connected to red point showing its position on $c_1(p)$ graph. Black line corresponds to c_1, red to c_2 and blue to c_3. All the trajectory plots have the same axis labels as plot (a).

trajectories – plots of time evolution of concentration $c_k(t)$ (see Fig. 3). We have chosen data for $q = 5$ as for this value we obtained the most intriguing results. Trajectories are presented together with data $c_1(p)$ for better understanding of their position on the phase diagram. Each red point in graph $c_1\,(p)$ corresponds

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to the given inset trajectory plot. For $p = 0.05$ and $c_2(0) = 1$ (panel (a) in Fig. 3) opinion $k = 2$ dominates for the whole simulation time. Trajectory (b) for $p = 0.09$ and $c_1(0) = 1$ shows similar character with domination of $k = 1$. Trajectory (c) for $p = 0.1$ and $c_1(0) = 1$ is qualitatively different from both previous plots. State $k = 1$ dominates for about 1000 MCS and then rapidly looses domination in favour of $k = 2$. Opinion $k = 3$ initially slowly increases, then the sudden rapid growth is observed but after very short time it drops to the final steady value. This surprising non-monotonic behaviour originates from the growth of neighbouring opinion $k = 2$. Later the system stabilizes with domination of $k = 2$. Trajectory (d) for $p = 0.125$ and $c_1(0) = 1$ has similar character to plot (c) but the change of domination happens much faster (~ 100 MCS). Trajectory (e) for $p = 0.142$ and $c_1(0) = 1$ has similar character to (d) but $k = 2$ becomes dominant much faster. Its final concentration is considerably smaller than in all previous cases. Plot (f) for $p = 0.45$ and $c_1(0) = 1$ shows very fast transition into domination of $k = 2$ on the lowest value of all cases: $c_2 \approx 0.43$.

Those results indicate once more that the first transition localized at $p \approx 0.1$ corresponds to the change of domination: from $k = 1$ to $k = 2$. The second transition at $p \approx 0.14$ corresponds to the loss of the overwhelming domination of any state. From this point with growing p the system evolves towards mixed state with equally numerous states $k = 1, 3$ and slightly more numerous middle state $k = 2$.

5 Discussion

When opinion is multi-state people are rather unlikely to change their opinion dramatically [6, 7]. In the model we express this in terms of bounded confidence. We analyzed the q-voter model with three-state opinions and bounded confidence on a complete graph. We took into account two types of the social response: conformity and independence. There are two phase transitions: the first from a certain order into a different order, and the second from an order into disorder. The first transition is discontinuous in all analyzed cases, while the second transition is discontinuous only for $q \geq 5$. The system shows dynamics that is qualitatively different than the corresponding system without bounded confidence [5].

References

1. Castellano, C.; Munoz, M. A.; Pastor-Satorras, R. *Phys. Rev. E* **2009**, *80*, 041129.
2. Jedrzejewski, A.; Sznajd-Weron, K. *Comptes Rendus Physique* **2019**, *20*, 244-261.
3. Chmiel, A.; Sznajd-Weron, K. *Phys. Rev. E* **2015**, *92*, 052812.
4. Martins, A.C.R. *Eur. Phys. J. B* **2020**, *93*, 1.
5. Nowak, B.; Ston, B.; Sznajd-Weron, K. *Sci. Rep.* **2021**, *11*, 6098.
6. Stauffer, D. *Advances in Complex Systems* **2002**, *5*, 97-100.
7. Hegselmann, R.; Krause, U. *JASSS* **2002**, *5*.
8. Alos-Ferrer, C.; Granic, D.-G. *Elect. Stud.* **2015**, *39*, 56-71.

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