PIES for viscoelastic analysis

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Abstract. The paper presents the approach for solving 2D viscoelastic problems using the parametric integral equation system (PIES). On the basis of Kelvin model the PIES formula in time differential form is obtained. As solving procedure the time marching is adopted, by introducing a linear approximation of displacements. The proposed approach, unlike other numerical methods, does not require discretization even the boundary. It uses curves as a tool for global modeling of boundary segments: curves of the first degree for linear segments and of the third degree for curvilinear segments. The accuracy is steered by the approximation series with Lagrange basis functions. Some test are made and shown in order to validate the proposed approach.

Keywords: PIES · Viscoelasticity · Time marching · Parametric curves.

1 Introduction

Many materials together with their elastic properties also show viscous characteristics. They require special treatment, because exhibit time-dependent strain. The two best known methods for solving elastic problems are the finite element method (FEM) [1] and the boundary element method (BEM) [2]. In the literature are also available various procedures for viscoelastic analysis, i.a. the one which transforms a viscoelastic equation into a pseudo-elastic [3] or uses an incremental scheme, where viscous behavior is added to the elastic responses [4]. An alternative to them is time marching process applied in both FEM and BEM [5, 6]. It bases on the differential constitutive relation for well-known viscoelastic models and allows for quick reaching steady states by speeding up time integration. Moreover, such approach makes easier changes in boundary conditions and viscous parameters along time. On the other hand it can influence the accuracy especially at small time scales. Referring to the FEM and BEM themselves, also in this approach they require spatial approximation by elements. In FEM the whole domain is divided into finite elements regardless of the problem solved. In BEM it depends on the formulation: one is defined in the domain and thus divides it into cells [5], while another transforms the problem to the boundary modeled by boundary elements [6].

The authors in their own research develop the approach that is an alternative to the above-mentioned element methods. The parametric integral equation system (PIES) [8–10] does not require traditional discretization, because the shape

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is analytically incorporated into its mathematical formalism. For this reason, various curves can be used to model the boundary [11], and their number is determined only by the shape, not by the accuracy expectations. It comes from the fact that in PIES the approximation of the boundary and the solutions are separated. The latter are approximated by special series, and the accuracy is controlled by changing the number of its terms. The efficiency of PIES has been confirmed on various examples from many fields like elasticity [8], acoustics [9] or plasticity [10]. In order to distinguish PIES from the recently popular isogeometric analysis [7] which also bases on CAD design tools, it should be emphasized that the curves used in PIES are integrated into it at the analytical level. As a result, any modification of the shape causes an automatic modification of the PIES formalism. Moreover, the isogeometric analysis very often requires elements, not for modeling, but for the integration. Finally, the approximation of the solutions and the shape in PIES are independent, which allows various approaches to be used for both.

The main aim of the paper is to obtain the PIES formula and the algorithm of its numerical solving for 2D viscoelastic problems. For the sake of simplicity, PIES is developed using the Kelvin viscoelastic model, but other models could be added to the approach following similar steps. The problem is defined only on the boundary, using Bezier curves for its modeling. The time marching methodology is applied using the linear approximation for the displacement time derivative. As a result, accurate solutions obtained by fewer computational resources are expected. The numerical examples included confirm assumptions in comparison to analytical solutions.

2 PIES for viscoelastic problems

2.1 Basic relations for Kelvin model

The Kelvin viscoelastic model is represented by a purely viscous damper and elastic spring connected in parallel [12]. Therefore, it can be stated that the strains in each component are the same, while the total stress is the sum of stresses in each component

$$\varepsilon_{ij} = \varepsilon_{ij}^{e} = \varepsilon_{ij}^{v}, \quad \sigma_{ij} = \sigma_{ij}^{e} + \sigma_{ij}^{v}, \tag{1}$$

where σ_{ij}, ε_{ij} indicate the stress and strain tensors, while superscripts e and v stands for elastic and viscous parts.

Stress tensors in (1) can be written in terms of strain components as follows

$$\sigma_{ij}^{e} = C_{ijlm}\varepsilon_{lm}, \quad \sigma_{ij}^{v} = \eta_{ijlm}\dot{\varepsilon}_{lm}, \quad \eta_{ijlm} = \gamma C_{ijlm}. \tag{2}$$

where C_{ijlm}, η_{ijlm} are elastic and viscoelastic tensors, which for isotropic materials can be written as a function of one another using a viscosity constant γ. Using (1) and (2), the general Kelvin constitutive relation is obtained

$$\sigma_{ij} = C_{ijlm}(\gamma\dot{\varepsilon}_{lm} + \varepsilon_{lm}). \tag{3}$$

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Introducing viscous effects to the global equilibrium equation of the body and neglecting the dynamic terms and the body forces results in

$$\sigma^e_{ij,i} + \sigma^v_{ij,i} = 0. \tag{4}$$

2.2 PIES formula

PIES for 2D viscoelastic analysis is obtained on the basis of the differential equilibrium equation (4) and takes the following form

$$\begin{aligned} &0.5\boldsymbol{u}_l(\bar{s}) + 0.5\gamma\dot{\boldsymbol{u}}_l(\bar{s}) = \\ &\sum_{j=1}^{n}\int_{s_{j-1}}^{s_j}\left\{\mathbf{U}^*_{lj}(\bar{s},s)\boldsymbol{p}_j(s) - \mathbf{P}^*_{lj}(\bar{s},s)\boldsymbol{u}_j(s) - \gamma\mathbf{P}^*_{lj}(\bar{s},s)\dot{\boldsymbol{u}}_j(s)\right\}J_j(s)ds, \end{aligned} \tag{5}$$

where $\boldsymbol{U}^*_{lj}(\bar{s},s)$, $\boldsymbol{P}^*_{lj}(\bar{s},s)$ are the displacement and traction boundary fundamental solutions, while $\boldsymbol{p}_j(s)$, $\boldsymbol{u}_j(s)$ and $\dot{\boldsymbol{u}}_j(s)$ are the displacement, traction and displacement time derivative components. The boundary in PIES is defined in 1D parametric reference system, where s,$\bar{s}$ are parameters and they are limited by $s_{l-1} \leq \bar{s} \leq s_l$, $s_{j-1} \leq s \leq s_j$ (s_{l-1} and s_{j-1} mark the start of lth and jth segments, and s_l and s_j their ends). $J_j(s)$ is the Jacobian, n is the number of boundary segments and $l, j = 1..n$.

$\mathbf{U}^*_{lj}(\bar{s},s)$ and $\mathbf{P}^*_{lj}(\bar{s},s)$ are presented explicitly in [8, 10]. They include analytically integrated shape of the boundary, which can be defined by any parametric curves e.g. Bezier curves of various degrees [11]. The choice of the appropriate representation depends on the complexity of the shape, and the modeling itself is reduced to defining an appropriate number of control points.

3 Shape modeling

In viscoelastic problems solved by BEM the geometry is defined in two different ways depending on the formulation. One approach bases on the idea of FEM and requires defining the domain, and hence the cells [5]. The process is similar to dividing the area into finite elements (Fig. 1a). The second formulation transforms the domain integral into the boundary by certain algebraic operations [6]. This makes the use of cells unnecessary (Fig.1b).

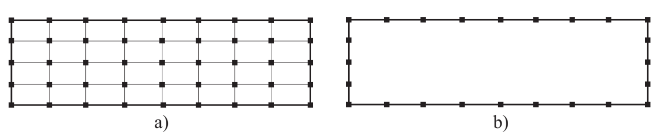

Fig. 1. The geometry modeled by: a) cells, b) boundary elements in BEM

Modeling the geometry in PIES is reduced to posing only control points. Their number and type depends on used curves. When linear segments of the

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boundary are defined, then only end points of each curve are specified (curves of the first degree). A rectangular shape created by corner points is presented in Fig. 2a. For curvilinear shapes, curves of the third degree (cubic) should be used. They require defining of four control points (two ends and two extra points responsible for the shape). An example of curvilinear geometry created by control points is shown in Fig. 2b.

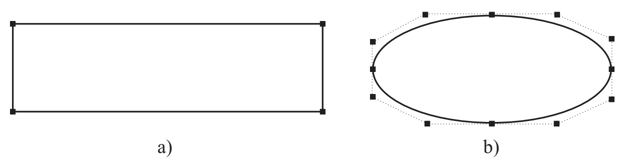

Fig. 2. The geometry modeled by: a) corner points, b) control points in PIES

In both mentioned cases in PIES the classical discretization is not required. The shape is modeled using the smallest number of data (e.g. corner points of the polygons), while the accuracy is provided by means of the appropriately selected number of coefficients in the approximating series (see the next section).

4 Solving procedure

Solving PIES (5) is reduced to finding unknown functions $\boldsymbol{u}_j(s)$, $\boldsymbol{p}_j(s)$ and $\dot{\boldsymbol{u}}_j(s)$. They are approximated by the following expressions

$$\boldsymbol{u}_j(s) = \sum_{r=0}^{R-1} \boldsymbol{u}_j^r L_j^r(s), \quad \boldsymbol{p}_j(s) = \sum_{r=0}^{R-1} \boldsymbol{p}_j^r L_j^r(s), \quad \dot{\boldsymbol{u}}_j(s) = \sum_{r=0}^{R-1} \dot{\boldsymbol{u}}_j^r L_j^r(s), \tag{6}$$

where $L_j^r(s) = \prod_{w=0, w\neq r}^{R} \frac{s-s_w}{s_r-s_w}$, $\boldsymbol{u}_j^r$, $\boldsymbol{p}_j^r$, $\dot{\boldsymbol{u}}_j^r$ are values of boundary functions at collocation points and R is the number of collocation points on jth segment.

After substituting (6) in (5), the PIES approximating form for the viscoelastic problems is obtained

$$0.5\boldsymbol{u}_l(\bar{s}) + 0.5\gamma\dot{\boldsymbol{u}}_l(\bar{s}) = \sum_{j=1}^{n}\sum_{r=0}^{R-1}$$
$$\left\{ \boldsymbol{p}_j^r \int_{s_{j-1}}^{s_j} \mathbf{U}_{lj}^*(\bar{s}, s) - \boldsymbol{u}_j^r \int_{s_{j-1}}^{s_j} \mathbf{P}_{lj}^*(\bar{s}, s) - \gamma\dot{\boldsymbol{u}}_j^r \int_{s_{j-1}}^{s_j} \mathbf{P}_{lj}^*(\bar{s}, s) \right\} L_j^r(s) J_j(s) ds. \tag{7}$$

Equation (7) after calculating all integrals and writing it at all collocation points takes the following shortened form

$$\boldsymbol{H}\boldsymbol{u} + \gamma\boldsymbol{H}\dot{\boldsymbol{u}} = \boldsymbol{G}\boldsymbol{p}, \tag{8}$$

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where $\boldsymbol{H} = [H]_{TT}$ and $\boldsymbol{G} = [G]_{TT}$ $(T = 2nR)$ are square matrices of elements expressed by integrals from (7), while $\boldsymbol{u}$, $\dot{\boldsymbol{u}}$, $\boldsymbol{p}$ are vectors containing coefficients of approximation series for displacements, displacement time derivatives and tractions.

In order to solve time differential equation (8) it is necessary to approximate velocity in time. It can be done by a simple linear time approximation as follows

$$\dot{\boldsymbol{u}}_{z+1} = \frac{\boldsymbol{u}_{z+1} - \boldsymbol{u}_z}{\Delta t}, \tag{9}$$

where Δt is the assumed time step. Applying (9) into (8) the following system of equations is obtained

$$(1 + \frac{\gamma}{\Delta t})\boldsymbol{H}\boldsymbol{u}_{z+1} = \boldsymbol{G}\boldsymbol{p}_{z+1} + \gamma \boldsymbol{H}\frac{\boldsymbol{u}_z}{\Delta t}. \tag{10}$$

The time marching process is obtained by solving (10) for each time step until the total time interval is reached, assuming that past values ($\boldsymbol{u}_z$) are known. The boundary conditions along time are prescribed by interchanging columns of $\boldsymbol{H}$ and $\boldsymbol{G}$.

5 Examples

The first example concerns a rectangular bar loaded at its free end. The material is under plane stress with the properties and other analysis parameters presented in Fig. 3a.

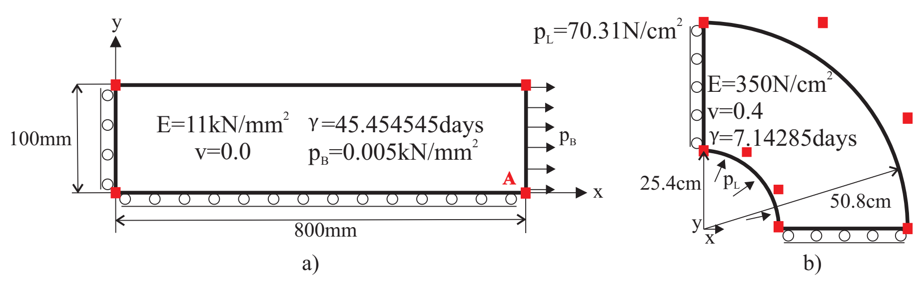

Fig. 3. The considered: a) stressed bar, b) thick cylinder

In PIES the boundary of the bar is modeled using 4 linear Bezier curves (4 corner points). It was also modeled by BEM in two ways. In [5] the domain of the bar is defined by 32 cells. Assuming they are linear, 45 nodes are used (not counting overlapping nodes). In the second approach [6], only the boundary is modeled using 48 boundary elements (48 nodes). In both cases, this is much more data to be modeled than in PIES.

The horizontal displacements at the bottom right corner of the bar (point A in Fig. 3a) are obtained at each time step ($\Delta t = 1$ day). Their comparison

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with analytical solutions is presented in Fig. 4. As can be seen obtained displacements are very similar to analytical results. Fig. 4 also presents dependence of the solutions on the time step length ($\Delta t = 1, 2, 5$ days). As mentioned in the introduction, the time marching can be less accurate at early times, which is visible for longer time steps.

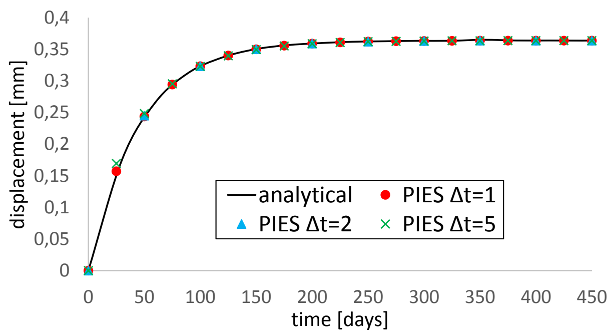

Fig. 4. The displacement at the bottom right corner of the bar (point A in Fig. 3a)

In the second example the behavior of a thick cylinder subjected to an internal pressure is analyzed. The geometry and physical properties are shown in Fig. 3b. The quarter of the cylinder in PIES is modeled by 2 cubic and 2 linear curves (8 control points). For comparison, in [5] BEM requires 12 cells (20 nodes).

The radial displacements of the outer boundary are obtained by PIES and compared with analytical solutions in Fig. 5. As can be seen, the results generated by PIES are very similar to analytical solutions, except some small discrepancies at early times.

Computational resources necessary to implement the proposed approach are also examined. The bar solutions are obtained within 0.137 second and the solved system of equations contains 32 equations. The quarter of the cylinder requires 0.174 second to be solved, with 48 equations in the system.

6 Conclusions

The paper presents the form of PIES for viscoelastic problems and the approach for its numerical solving. The shape is defined by parametric curves globally, i.e. without traditional discretization. Unknown displacements and tractions are approximated by series with Lagrange basis functions, while the displacement time derivative is obtained by linear approximation. The system of algebraic equations is solved for each time step until the predetermined period of time has elapsed.

The proposed approach is validated on two examples. A high agreement with the analytical results is obtained with much lower effort related to modeling of

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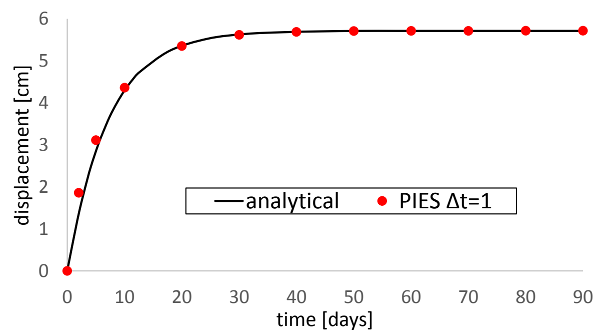

Fig. 5. The radial displacement on the outer boundary

the geometry. Moreover, fewer time and memory resources are required to obtain such solutions.

The presented technique can be generalized to 3D viscoelastic problems using parametric surfaces for boundary representation. Moreover, it can be helpful for solving viscoplastic problems with surfaces used for modeling the domain. However, this requires additional research and testing.

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