High Resolution TVD Scheme based on Fuzzy Modifiers for Shallow-Water equations

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Abstract. This work proposes a new fuzzy logic based high resolution (HR), total variation diminishing (TVD) scheme in finite volume frameworks to compute an approximate solution of the shallow water equations (SWEs). Fuzzy logic enhances the execution of classical numerical algorithms. To test the effectiveness and accuracy of the proposed scheme, the dam-break problem is considered. A comparison of the numerical results by implementing some classical flux limiting methods is provided. The proposed scheme is able to capture both smooth and discontinuous profiles, leading to better oscillation-free results.

Keywords: Shallow water equation · Fuzzy Modifier · Limiter.

1 Introduction

Shallow-water equations are frequently employed in the situations which involve the modelling of water flow corresponding to various water bodies such as lakes, rivers, reservoirs, and other such variants in which the fluid depth metric is significantly smaller than the horizontal length metric [14, 15, 21]. The standard SWEs (also known as the Saint Venant equations), were initially introduced about one and a half century ago and still these equations are used in various applications [13, 11, 12]. For many practical, real-life models, such as dam-break problems, flood problems, etc., these equations are frequently used. The solutions to such systems are generally non-smooth and produce discontinuities also, so it is essential to have a robust, efficient, and accurate numerical strategy for the Shallow water system and related models. Finite-volume schemes are among the most popular tools to tackle such situations.
The particular case in which SWEs are inviscid in nature, lead to an hyperbolic system of equations, and all the robust numerical strategies [10, 9] that have been constructed for hyperbolic conservation laws can be implemented to such equations. From a mathematical standpoint, the hyperbolic equations are well known to admit discontinuous solutions, and their numerical integration is expected to compute such discontinuities sharply and without oscillations. Based on the classical HR-TVD flux limiting schemes, this work addresses a new hybrid

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flux limiting method [22, 17, 7], which is used in this work to approximate the flow components at the midpoints of cell edges inside the control volumes of a computational domain. Hyperbolic conservation laws govern SWEs. By defining the SWEs with this broader class of equations, the opportunity to exploit the set of techniques and mathematical tools previously established for these computationally complicated situations opens up [14, 13, 12].
Motivation behind fuzzy-logic based approach: The fuzzy logic theory has evolved in a number of ways since Zadeh's introduction of fuzzy set theory. Fuzzy logic theory is now commonly used in fluid mechanics, control engineering, information processing, artificial intelligence, strategic planning, and other fields[6, 5]. Fuzzy control problems have made significant progress in recent decades being one of the most popular frameworks of fuzzy sets and fuzzy logic. The fuzzy-logic-based control has been commonly used in machine engineering, intelligence control, system recognition, image classification, neural networks, and other areas. In contrast to traditional crisp control, fuzzy logic controller will more accurately model physical reality in a linguistic format, allowing for more efficient method of achieving intelligent management in engineering settings. In Fuzzy mathematics, the concept of fuzzy logic is quite unique as compared to the classical logic, as fuzzy logic works more like the human way of reasoning. In other words, fuzzy logic approach is more easy and understandable. Fuzzy logic has many applications in almost are the industries related to various commercial and practical purposes. In artificial intelligence, Fuzzy Logic helps in simulating the human oriented cognitive processes.
The hybrid method's main merit is its optimized construction using an entirely different concept from fuzzy logic [5], which makes it better than the classical limiters. The optimized fuzzy flux limited scheme is implemented into a one-dimensional structured finite volume model to approximate the shallow water flows. This work concentrates on the optimization of classical numerical methods for observing the behavior of Dam-Break Problem governed by one-dimensional shallow water equation in which discontinuities are present and are important to model. The computational results of the SWEs with the proposed scheme is assessed by experimenting with the dam-break problem [4, 8]. The proposed scheme results are compared with those obtained from the classical minmod scheme and the monotonized-central (MC) scheme for validation.
The work is further structured as follows. The numerical ow model is explained in the Section 2. In the Section 3, the hyperbolic numerical approach is explained. In the frame of uniform mesh and finite volume methods, the proposed new scheme with a brief discussion on fuzzy logic cocepts is introduced in the Section 4. After that, a numerical assessment is shown in the Section 5. Further, the work is concluded with some remarks and the scope of future work in the Section 6.

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2 Numerical model: One-dimensional Shallow water Flows

In one space dimension, the SWEs [1] can be described in mathematical form as:

$$q_t + f(q)_x = s, \tag{1}$$

where

$$q = \begin{pmatrix} h \\ hu \end{pmatrix}, \; f(q) = \begin{pmatrix} hu \\ hu^2 + \frac{1}{2}gh^2\alpha \end{pmatrix}, \; s(q) = \begin{pmatrix} 0 \\ -ghZ_x \end{pmatrix}.$$

Here, the height of water is denoted as $h(x,t)$, the fluid velocity as $u(x,t)$, the notation for acceleration due to gravity is g and the bottom surface function is denoted by $Z(x)$. As, the present work is concerned towards hyperbolic conservation laws, so this function $Z(x)$ is taken to be zero. So, the Equation 1 becomes:

$$\partial_t \begin{pmatrix} h \\ hu \end{pmatrix} + \partial_x \begin{pmatrix} hu \\ hu^2 + \frac{1}{2}gh^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{2}$$

In this work, the finite-volume framework is used as it prevents any global transformation in the conserved variables, so the overall scheme remains conservative in nature. Here as illustrated in the upcoming sections, both space and temporal discretizations are done in a higher-order accurate manner. A spatial reconstruction, called the MUSCL (Monotone Up-stream Centered Scheme for Conservation Laws) technique, has been considered to obtain higher order accuracy in space.

3 Numerical approach: Flux limiting High Resolution schemes

To formulate the finite-volume framework, the primary task is to discretize the space domain in forms of smaller cells $[x_{i-1/2}, x_{i+1/2}]$ (refer to the Figure 1), which have a uniform spatial step of length Δx, such that $x_i = \Delta x(i + 1/2)$. Similarly, the temporal domain is discretized into sub intervals $[t^n, t^{n+1}]$ with uniform step size Δt, such that $t^n = \Delta t(n)$. $[x_{i-1/2}, x_{i+1/2}]$ denotes the i^{th} control volume, where $x_i = (i+1/2)\Delta x$ is the mid-point of this control volume. A numerical integration of the nonlinear conservation laws, discussed in the Section 2 requires a finite volume Godunov method of upwind-type. A conservative form related to the homogeneous equation 1 is written as:

$$q_i^{n+1} = q_i^n - \frac{\Delta t}{\Delta x}\Big(F_{i+\frac{1}{2}} - F_{i-\frac{1}{2}}\Big), \tag{3}$$

where

$$q_i^n \approx \frac{1}{\Delta x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} q(x,t^n)dx \tag{4}$$

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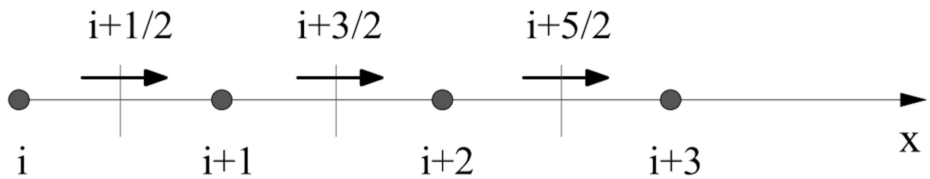

Fig. 1: A spatial representation of the computational grid.

is the cell average corresponding to the spatial components, and $F_{i\pm1/2}$ denote the numerical flux functions (temporal cell-averages), defined in the following manner:

$$F_{i-\frac{1}{2}} \approx \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} f(q(x_{i-\frac{1}{2}},t))dt. \tag{5}$$

The important task in approximating such conservation laws is the proper selection of the flux presented in the Equation 5. In general, this construction demands the solutions of various Riemann problems at the cell interfaces.
In a finite-volume HR technique, the numerical flux is calculated by mixing a lower and a higher order flux altogether. For $f(q) = a(q)$ with positive speed a, the mathematical form of the Lax-Wendroff technique is:

$$q_i^{n+1} = q_i^n - \nu(q_i^n - q_{i-1}^n) - \frac{1}{2}\nu(1-\nu)(q_{i+1}^n - 2q_i^n + q_{i-1}^n), \tag{6}$$

where $\nu = a.\Delta t/\Delta x$ is termed as the Courant number. It is an upwind scheme of first-order along with an extra anti-diffusive term of second-order. The scheme in the Equation 6 is second-order accurate, but it still does not follows the TVD property. Therefore, the Equation 6 is further modified by introducing a limiting function, say, ϕ to the term of second order in the following manner:

$$q_i^{n+1} = q_i^n - (q_i^n - q_{i-1}^n)\left[\nu + \frac{1}{2}\nu(1-\nu)\left(\frac{\phi(r_{i+\frac{1}{2}})}{r_{i+\frac{1}{2}}} - \phi(r_{i-\frac{1}{2}})\right)\right], \tag{7}$$

where the function $r_{i+1/2}$ is defined as:

$$r_{i+\frac{1}{2}} = \frac{q_i^n - q_{i-1}^n}{q_{i+1}^n - q_i^n}. \tag{8}$$

A HR scheme is developed when the limiting function given in the Equation 8 is positive [3]. To advance the solution in time, the numerical fluxes are calculated as follows:

$$F(q_{i+\frac{1}{2}}) = f_{i+\frac{1}{2}}^l - \phi(r_i)\big(f_{i+\frac{1}{2}}^l - f_{i+\frac{1}{2}}^h\big), \tag{9}$$

here, f^l resembles the low-resolution and f^h resembles the high-resolution [17] numerical flux functions.

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Theorem 1 (Harten's Lemma). *A numerical method can be formulated in the incremental form as:*

$$q_i^{n+1} = q_i^n - C_{i-1/2}^n \Delta q_{i-1/2}^n + D_{i+1/2}^n \Delta q_{i+1/2}^n. \tag{10}$$

If $\forall n \in Z$, *and each integral value* i, *the coefficients follow the constraints presented as follows:*

$$C_{i+1/2}^n \geq 0, \tag{11}$$

$$D_{i+1/2}^n \geq 0, \tag{12}$$

$$C_{i+1/2}^n + D_{i+1/2}^n \leq 1, \tag{13}$$

then such a numerical scheme is TVD.

For the implementation of flux limiters [11] in the numerical scheme, the reconstruction step should obey an additional TVD property given as:

$$\phi_{sweby}(r) = \max\{0, \min\{2r, 2\}\}. \tag{14}$$

Table 1 gives a quick introduction to some of the commonly used flux limiters.

Table 1: Some TVD flux limiting functions.

Limiter	Representation	Remarks
Minmod	$max(0, min(r, 1))$	Roe, 1986 [20]
Superbee	$max(0, min(2r, 1), min(r, 2))$	Roe, 1986 [20]
Van Albada	$\frac{r(r+1)}{r^2+1}$	Van Albada, et al., 1982 [3]
Monotonized Central	$max(0, min(min(2r, (1+r)/2, 2)))$	Van Leer, 1977 [11]
Van Leer	$\frac{r+\|r\|}{1+\|r\|}$	Van Leer, 1974 [21]

For a detailed theory refer to the citations provided with each limiter in the Table 1. Further, a graphical representation of these flux limiters is shown in the Sweby's TVD region [19], as seen in the Figure 2. The present work highlights a fundamental idea to modify and optimize the classical limiters to enhance the overall numerical outputs.

4 Development of the New Flux Limiter scheme

The present algorithm consists of optimizing a specic classical ux limiter to form a better hybrid alternative. Many flux limiters are available in the literature to prevent discontinuities [7]. To optimize the classical schemes, some important concepts from the literature of Fuzzy mathematics are also required.

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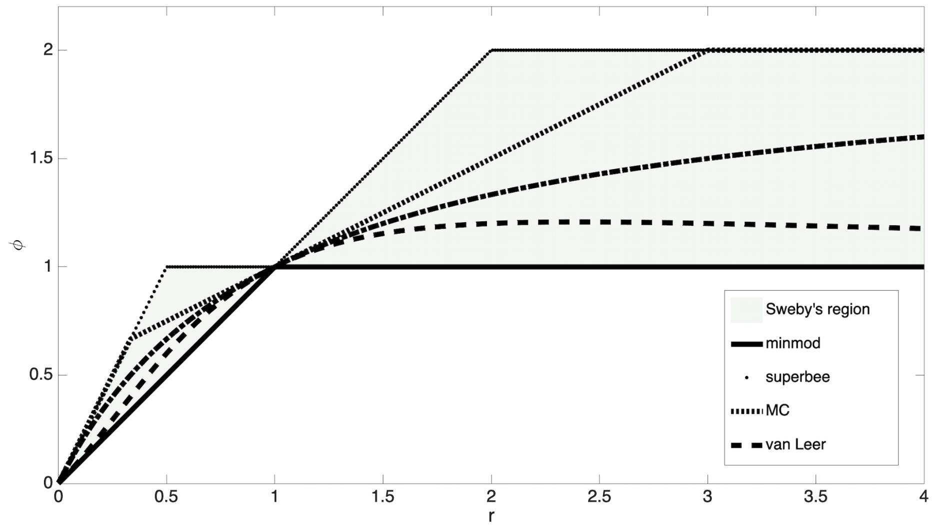

Fig. 2: Graphical representation of the classical Flux Limiters mentioned in the Table 1.

Fuzzy sets: A fuzzy set is basically a classical (crisp) set having a special property, which allows each member of the considered universal set to get connected with this crisp set by a suitable intensity (called membership value). The membership intensity depends on the degree of compatibility of a particular element with the crisp set. The most commonly used set for membership degrees in fuzzy sets is $[0, 1]$, however this set restricts to the discrete values $\{0, 1\}$ for a crisp set. Mathematically, for a classical set T in the universe of discourse U, a fuzzy set A could be presented as follows: $A = \{(x, \mu(x)) \mid x \in T\}$, where the membership function μ sends the members of the classical set T to the closed interval $[0, 1]$.

Fuzzy Linguistics: These are known as the fuzzy variables originated through a special domain consisting of words, it's members are also known as the linguistic entities in the context of Fuzzy mathematics. These variables help to associate the elements of the universal set with a suitable membership value, using which a relationship of that element could be defined with the concerned fuzzy set [5]. As fuzzy values capture measurement uncertainties as a consequence of initial data sets, these are much more adaptive than the crisp variables to real-life models.

Fuzzy Modifiers: Fuzzy modifiers are an important ingredient in the construction of the new limiter. Fuzzy hedges fine tune the interpretation of the given data by modifying the membership units for the related fuzzy sets. Corresponding to the fuzzy set A defined above, several commonly used fuzzy hedges are: the Dilation modifier $(\{(y, \sqrt[p]{(\mu(y))}) \mid y \in U\})$, the Concentration modifier $(\{(y, (\mu(y))^p)) \mid y \in U\})$ [12], here p is some arbitrary real value.
Further this section comprises of enhancing the classical limiters to establish a

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new limiter. The procedure in this section is centered on how to use suitable fuzzy modifier operations to fine-tune parameter settings in the classical flux-limited schemes.

Formulation of the new hybrid flux limiter: For this work, an optimization of the most commonly used monotonized central (MC) limiter is shown. The MC limiter in the sense of a piecewise function is:

$$\phi(r) = \begin{cases} 0, & r \leq 0 \\ 2r, & 0 < r \leq 1/3 \\ \frac{1}{2}(1+r), & 1/3 < r \leq 3 \\ 2, & \text{else.} \end{cases} \tag{15}$$

This is improvised by assigning the concentration modifier function of intensity $p = 6$ and $p = 8$ to the smooth and extrema regions respectively, and other parts are kept the same [17]. So, the hybrid limiter turns out to be:

$$\phi(r) = \max\left(0, \min\left(\min\left(\frac{\frac{2}{3}(\frac{9-3r}{8})^6 + 2\frac{3r-1}{8}}{(\frac{9-3r}{8})^6 + \frac{3r-1}{8}}, 2, \frac{\frac{2}{3}(3r)^6}{(1-3r)+(3r)^6}\right)\right)\right). \tag{16}$$

The hybrid flux limiter is shown in the Figure 3. This procedure opens up infinitely many choices for the flux limiter functions in the context of Fuzzy Mathematics [18, 22]. To show the performance of the hybrid fuzzy limiter presented

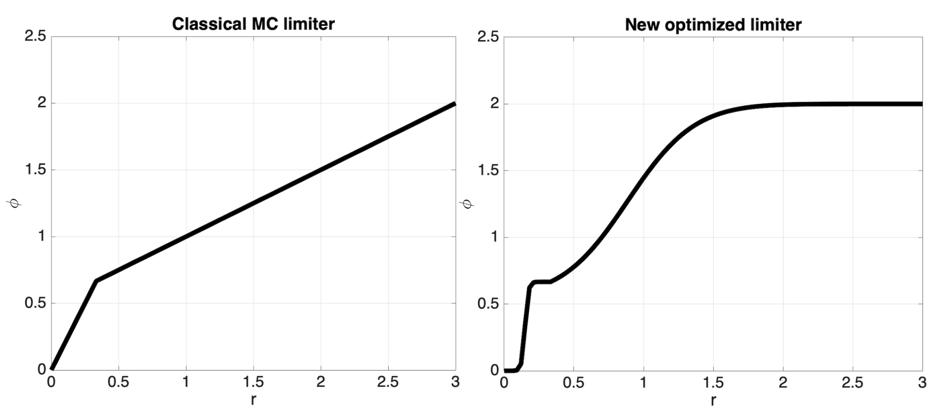

Fig. 3: The hybrid fuzzy flux limiter.

in the Equation 16, the Shallow water problem, written as the Equation 2 is approximated in the upcoming section.

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5 Numerical Validation: Using Dam break problem

For the numerical computations in this section, the classical limiter functions: the Min-Mod and the MC limiters have been utilised to compare the approximate results obtained from the proposed hybrid limiter. The flux limiting techniques are based on a deterministic finite volume solver. For computational puposes, MATLAB 2015b version has been used with macOS Mojave, RAM 8 GB and 2.3 GHz Intel Core i5.
Throughout this section of numerical validation, the acceleration due to gravity is set to $g = 9.81$ and the standard SI measuring units corresponding to the physical quantities (like s (seconds), kg (kilograms), m (meters), etc.) are omitted in the discussion. For the space discretization, we have used uniform cartesian grids.
To analyse the performance of the hybrid method discussed in this paper, let q_i^n be the numerically computed solution and $q(x_i, t^n)$ be the exact solution corresponding to the i^{th} control volume at the n^{th} time stamp, thus the L_1 error norm, presented here by $\|e_n\|_1$ is written as:

$$\|e_n\|_1 = \sum_{i=1}^{N} |q(x_i, t^n) - q_i^n| \, \Delta x. \tag{17}$$

and the L_∞ error norm, denoted by $\|e_n\|_\infty$ is given as follows:

$$\|e_n\|_\infty = \max_{1 \le i \le N} |q(x_i, t^n) - q_i^n| \, . \tag{18}$$

where N represents the computational points.

5.1 Dam break

In this section, the shallow problem (2) is computed to assess the proposed solution scheme by using various test cases corresponding to the dam-break scenario in a rectangle shaped domain having flat topography (i.e., $Z(x) = 0$, refer 1). The computational domain is $[-1, 1]$, and the step size is $\Delta x = 0.005$. In the next subsection, three test cases have been considered. The test case 1 corresponds to the Riemann problem in height profile, the test case 2 is basically opposite of the test case 1, and the third test case represents a vacuum Riemann problem.

Test case 1: The initial profile used for approximating this test situation of the dam break problem is:

$$u(x, t = 0) = 0; \; h(x, t = 0) = \begin{cases} 0.1, & -1 < x \le 0 \\ 2, & \quad 0 < x \le 1. \end{cases} \tag{19}$$

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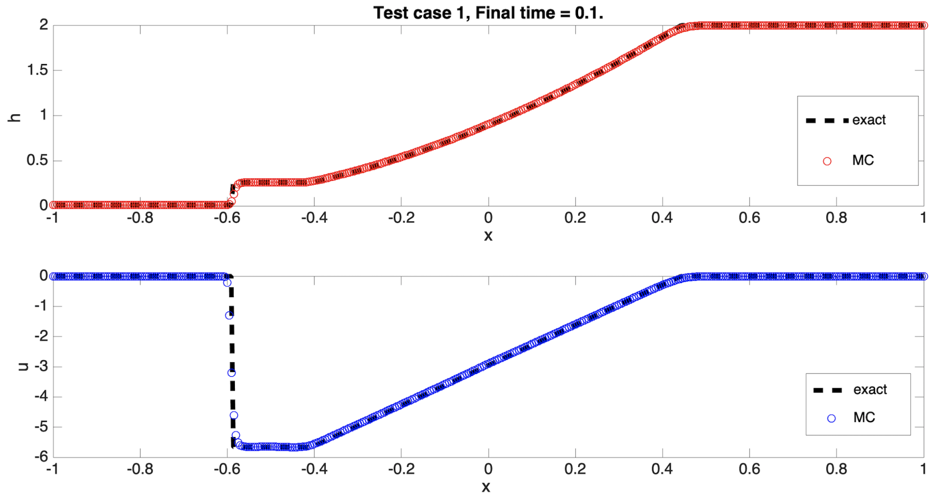

Fig. 4: Approximation results corresponding to the test case 1 obtained by the classical limiter for $N = 400$ control volumes at final time $t = 0.1$.

Table 2: Error Analysis based on L_1 norm for the test case 1

Final Time	Minmod (MM)	Monotonized-Central (MC)	Proposed (New)
0.1	1.40e-03	1.12e-03	8.36e-04
0.05	1.36e-03	1.09e-03	1.02e-03
0.02	1.63e-03	1.45e-03	1.37e-03

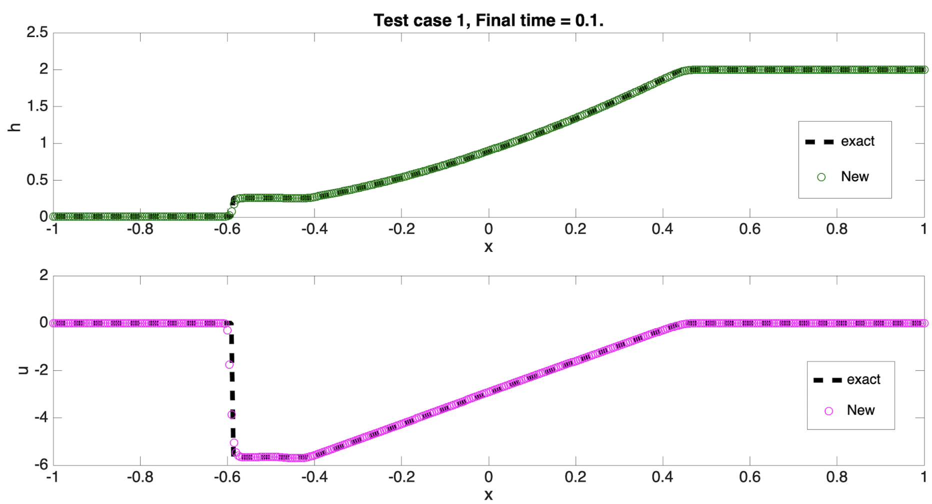

Fig. 5: Approximation results corresponding to the test case 1 obtained by the proposed limiter for $N = 400$ control volumes at final time $t = 0.1$.

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Table 3: Error Analysis based on L_∞ norm for the test case 1

Final Time	Minmod (MM)	Monotonized-Central (MC)	Proposed (New)
0.1	2.13e-01	1.86e-01	1.11e-01
0.05	1.95e-01	1.66e-01	1.05e-01
0.02	2.01e-01	1.97e-01	1.00e-01

Tables 2 - 3 provide the details of point-wise error analysis for the standard flux limiting functions and the proposed method for the L_1 and the L_∞ norms, and the Figures 4-5 present the computational results corresponding to the classical MC limiter and the proposed limiting function for 400 computational points.
It is visible from the Figure 4 that the MC limiting function is able to grasp the solution profile, although slight oscillations can still be seen. However, the computational output appears to be improved for the hybrid limiter, as seen in the Figure 5.

Test case 2 The following is the initial data profile for simulating this dam-break test case:

$$u(x,t=0)=0;\; h(x,t=0)=\begin{cases} 2, & -1<x\le 0 \\ 0.1, & 0<x\le 1. \end{cases} \tag{20}$$

The MC limiter is clearly able to capture the solution structure, as shown in the Figure 6, though minor perturbations can still be seen. Nevertheless, as shown in the Figure 7, the numerical result for the hybrid limiter appears to be improved.

Test case 3: The initial data for approximating this dam-break test case is:

$$h(x,t=0)=0.1;\; u(x,t=0)=\begin{cases} -2, & -1<x\le 0 \\ 2, & 0<x\le 1. \end{cases} \tag{21}$$

Although slight disturbances can still be seen in the solution pattern, but the MC limiter is clearly able to capture it, as shown in the Figure 8. However, as can be seen in the Figure 9, the numerical result for the hybrid limiter appears to be better.

5.2 CPU Time

The comparison of results obtained by various test cases has been the primary focus in the Subsection 5.1. In terms of the CPU time taken by the various

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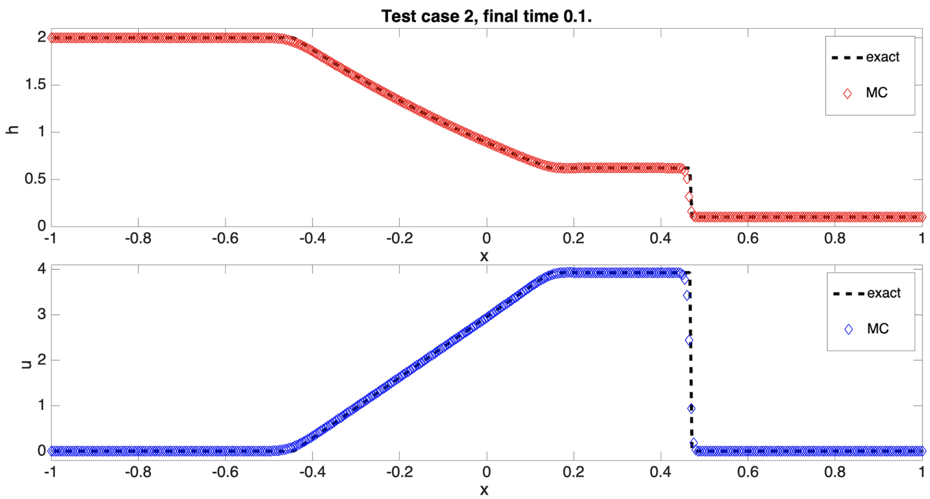

Fig. 6: Approximation results corresponding to the test case 2 obtained by the classical limiter for $N = 400$ grid points at final time $t = 0.1$.

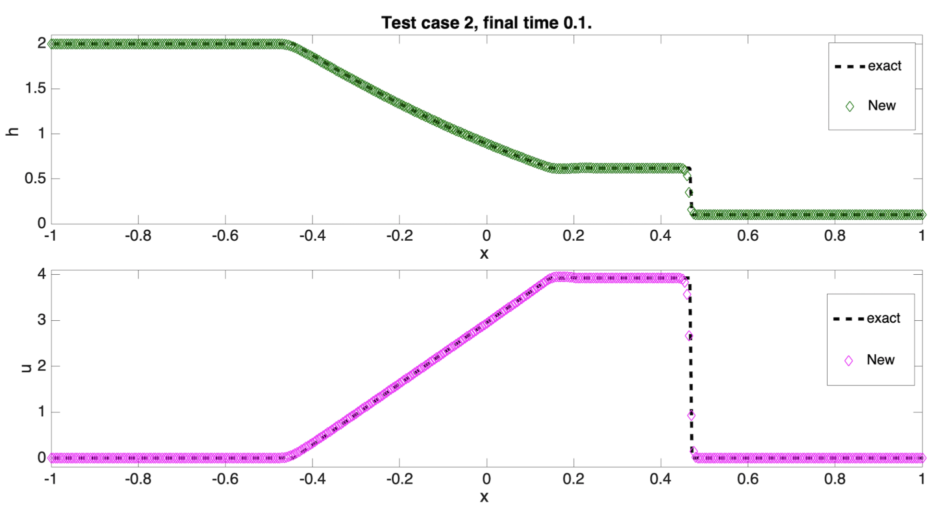

Fig. 7: Approximation results corresponding to the test case 2 obtained by the proposed limiter for $N = 400$ grid points at final time $t = 0.1$.

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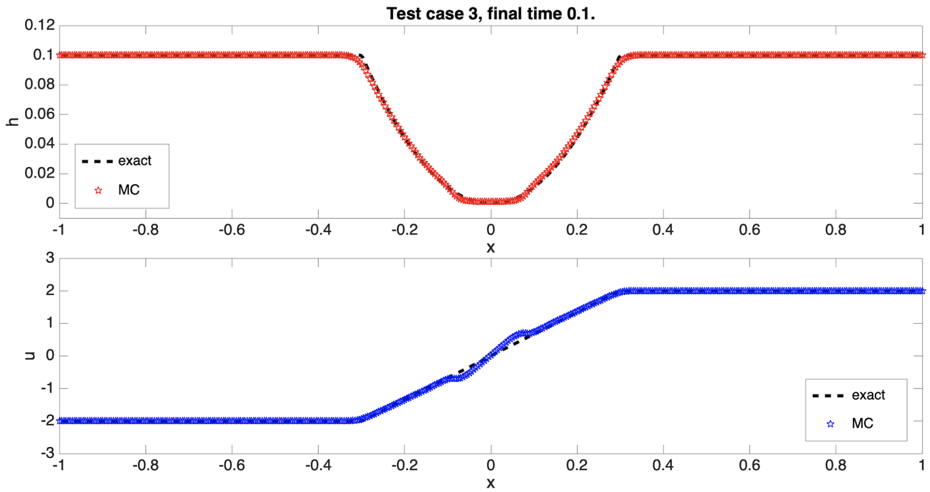

Fig. 8: Approximation results corresponding to the test case 3 obtained by the classical limiter for $N = 400$ grid points at final time $t = 0.1$.

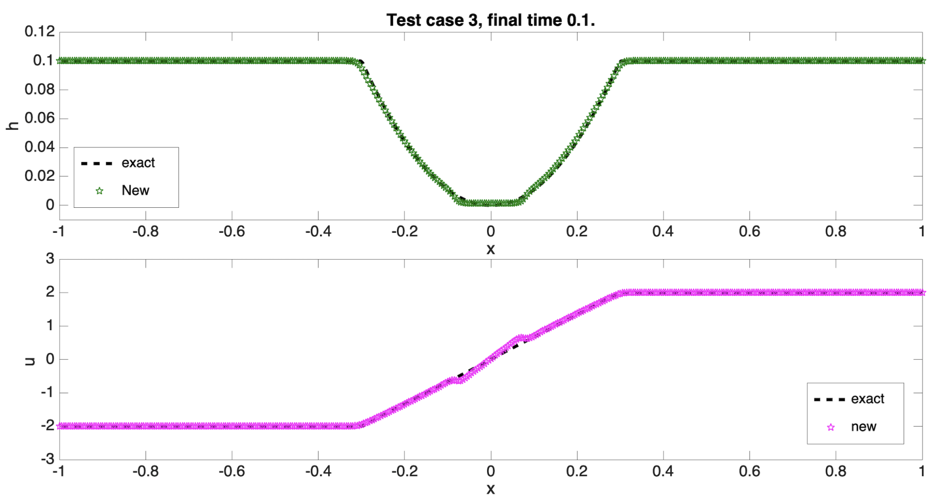

Fig. 9: Approximation results corresponding to the test case 3 obtained by the proposed limiter for $N = 400$ grid points at final time $t = 0.1$.

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numerical integrations, the proposed scheme requires relatively more CPU time in all of the test cases since the number of operations per time step involved in calculating the fluxes across neighboring cells is greater than the standard flux limiting schemes. Refer to the Table 4 for the comparison of CPU times for various test cases considered in the Subsection 5.1.

Table 4: CPU time data in seconds (s)

CPU Time	Test case 1	Test case 2	Test case 3
Monotonized-Central (MC)	2.08644s	2.11922s	1.40411s
Proposed (New)	2.40097s	2.47627s	1.71645s

6 Conclusion

In summary, we have presented a numerical formulation of SWEs using a fuzzy logic based flux limiting scheme. The approach is based on the physical principles and balance laws of classical fluid mechanics. The distinction between the classical and the proposed method lies in its main new ingredient: Fuzzy modifiers. As future work, this strategy could be further extended to higher dimensional SWEs. The proposed HR method is non-oscillatory, conservative, well balanced, and suitable for shallow water models. The proposed flux limited method is verified against the benchmark dam-break problem with flat bottom topography. Final results are comparable with the classical scheme and show good agreement with analytical solutions also.

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