Comprehensive regularization of PIES for problems modeled by 2D Laplace's equation

Krzysztof Szerszeń [0000-0001-9256-2622], Eugeniusz Zieniuk [0000-0002-6395-5096], Agnieszka Bołtuć [0000-0002-4761-1170] and Andrzej Kużelewski [0000-0003-2247-2714]

University of Bialystok, Institute of Computer Science
15-245 Białystok, Konstantego Ciołkowskiego 1M, Poland
{kszerszen, ezieniuk, aboltuc, akuzel}@ii.uwb.edu.pl

Abstract. The paper proposes the concept of eliminating the explicit computation of singular integrals appearing in the parametric integral equation system (PIES) used to simulate the steady-state temperature field distribution. These singularities can be eliminated by regularizing the PIES formula with the auxiliary regularization function. Contrary to existing regularization methods that only eliminate strong singularities, the proposed approach is definitely more comprehensive due to the fact that it eliminates all strong and weak singularities. As a result, all singularities associated with PIES's integral functions can be removed. A practical aspect of the proposed regularization is the fact that all integrals appearing in the resulting formula can be evaluated numerically with a standard Gauss-Legendre quadrature rule. Simulation results indicate the high accuracy of the proposed algorithm.

Keywords: computational methods, regularized PIES, singular integrals, potential problems 2D, Bézier curves

1 Introduction

One of the most significant problems to be faced during computer implementation of parametric integral equation system (PIES) is the evaluation of singular integrals. Their presence is related to the fact that PIES is based on the analytical modification of the conventional boundary integral equation (BIE). This modification, previously presented for various types of differential equations [1-3], is aimed to include analytically the shape of the boundary problem directly in the obtained PIES formula. As a result, in opposite to finite and boundary element methods (FEM, BEM), PIES's solutions of boundary value problems are obtained without domain or boundary discretization. Finally, we can introduce some alternative representations of the boundary shape. In the case of 2D problems, it is particularly promising to define the boundary with parametric curves, e.g. Bézier curves. Hence, instead of a mesh of boundary elements with their nodes, we can use parametric curves of different degrees defined by a relatively small set of control points. Moreover, the obtained formal separation between the declared boundary and boundary functions in PIES allows to approximate the boundary functions by effective Chebyshev series.

ICCS Camera Ready Version 2021
To cite this paper please use the final published version:
DOI: 10.1007/978-3-030-77961-0_38

The presence of singular integrals is a common problem both for BIE and PIES methods and is related to their integral kernels dependent on the distance between the so-called source and field points. In the case when these points are close to each other, this distance tends to zero and the kernels are singular. The degree of singularity depends on the form of the kernel functions. Accurate evaluation of singular integrals plays a crucial role in the overall accuracy of solutions of boundary value problems. This is even more difficult because the direct evaluation of singular integrals by the popular Gauss-Legendre (G-L) quadrature may result in an unacceptable accuracy degradation. Due to the importance of the problem, there is an extremely rich literature on this subject with many algorithms for evaluation of the singular integrals. They are mainly dedicated to BEM, among which we can mention nonlinear transformations [4-6], adaptive subdivision [7], variable transformation [8], semi-analytical methods [9,10], as well as quadrature methods [11]. One of the most promising are regularization methods [12-15].

This paper proposes a new algorithm to eliminate weakly and strongly singular integrals in PIES. The algorithm is based on the regularization of the PIES formula using the auxiliary regularizing function with regularization coefficients. As a result, all obtained regularized integrals are no longer singular and can be evaluated by the standard G-L quadrature. To demonstrate the capability and accuracy of the proposed scheme we present two simulation examples.

2 The singular formulation of the PIES

The paper deals with the prediction of the steady-state temperature field distribution inside the 2D domain Ω and on the boundary Γ. The model formulation is based on the linear boundary value problem for the Laplace equation

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} = 0, \tag{1}$$

with Dirichlet u_Γ and Neumann p_Γ boundary conditions, as shown in Fig. 1a.

In the case of practical problems defined for more complex geometries and boundary conditions, we need to use numerical computational methods, for example finite element method (FEM) or boundary element method (BEM). Fig. 1b shows modeling of the domain Ω by FEM with finite elements. A similar discretization strategy is related to BEM, but refers only to the boundary Γ as shown in Fig. 1c. Such modeling has gained wide popularity, but in practice it requires to generate a significant number of elements as well as algebraic equations.

ICCS Camera Ready Version 2021
To cite this paper please use the final published version:
DOI: 10.1007/978-3-030-77961-0_38

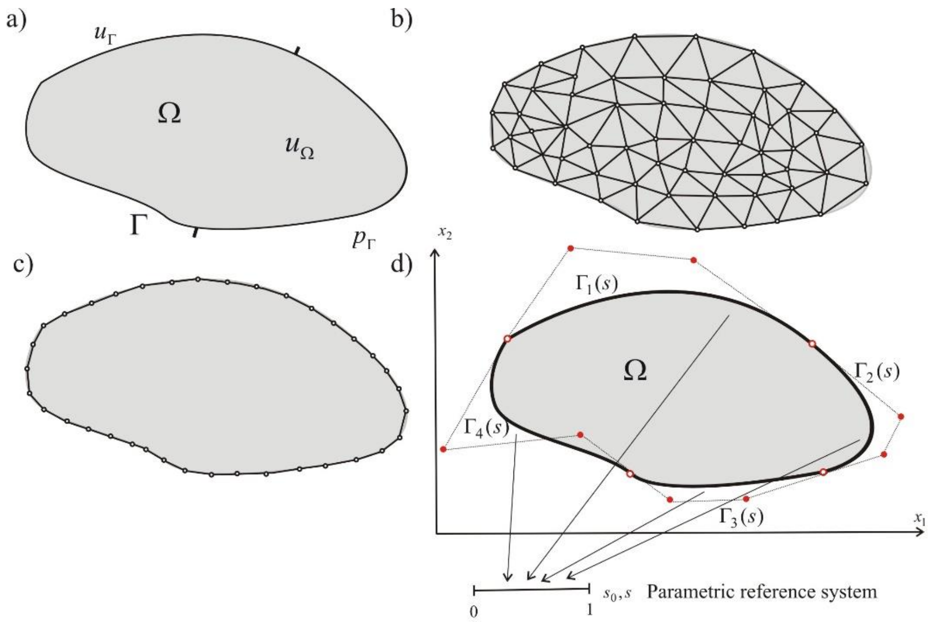

Fig. 1. Problem's domain, boundary shape and boundary conditions (a), modeling of a 2D domain with finite elements (b) and its boundary with boundary elements (c), definition of the boundary with Bézier curves in relation to the parametric reference system in PIES (d).

Here, to overcome some of the limitations observed in the case of FEM and BEM, an alternative approach for solving boundary value problems, called as PIES, is used. The PIES has several advantages and we can identify them as:

- The prediction of the temperature field in the interior of the domain, similar to BIE, is obtained via the analysis of the problem only on the boundary of that domain. This reduces the mathematical dimension of the problem under analysis by one.
- Only the boundary of the domain needs to be defined. The boundary in PIES is parameterized and described in a very general way as a closed parametric curve (e.g. Bézier and NURBS curves). Fig. 1d shows practical definition of the boundary from Fig. 1a by joining 4 Bézier curves of degree 3. Bézier curves allow for intuitive description of the boundary using only control points and it is more effective than the classical discretization with boundary elements in BEM.
- The boundary description with Bézier curves is analytically included into the PIES formula used to find solutions on the boundary of the problem. For the Laplace equation this formula is written as [2]

$$0.5u_l(s_0) = \sum_{j=1}^{n} \int_0^1 \left\{ \overline{U}_{lj}^*(s_0,s)p_j(s) - \overline{P}_{lj}^*(s_0,s)u_j(s) \right\} J_j(s)ds, \; s_0, s \in \langle 0,1 \rangle . \quad (2)$$

  Formula (2) is not specified directly on the boundary, as is the case of classical BIE, but on a straight line representing a projection of boundary segments represented by by n Bézier curves into parametric reference system dependent on parameter s, as

ICCS Camera Ready Version 2021
To cite this paper please use the final published version:
DOI: 10.1007/978-3-030-77961-0_38

shown in Fig. 1d. Moreover s_0 is the co-called source point located in the same parametric reference system and $J_j(s)$ is the Jacobian of transformation from Cartesian to parametric coordinates. The Jacobian is determined for all Béziera curves by the following formula

$$J_j(s) = [\left(\frac{\partial \Gamma_j^{(1)}(s)}{\partial s}\right)^2 + \left(\frac{\partial \Gamma_j^{(2)}(s)}{\partial s}\right)^2]^{0.5}. \tag{2a}$$

- Integral kernels in (2) represented in the form

$$\bar{U}_{lj}^{*}(s_0,s) = \frac{1}{2\pi}\ln\frac{1}{[\eta_1^2+\eta_2^2]^{0.5}}, \quad \bar{P}^{*}_{lj}(s_0,s) = \frac{1}{2\pi}\frac{\eta_1 n_j^{(1)}(s)+\eta_2 n_j^{(2)}(s)}{\eta_1^2+\eta_2^2}, \tag{3}$$

  include analytically in their mathematical formalism the boundary shape generated by Bézier curves by following relations

$$\eta_1 = \Gamma_l^{(1)}(s_0) - \Gamma_j^{(1)}(s), \ \ \eta_2 = \Gamma_l^{(2)}(s_0) - \Gamma_j^{(2)}(s), \tag{4}$$

  where functions $\boldsymbol{\Gamma}_l\,(s_0)$, $\boldsymbol{\Gamma}_j(s)$ describe Bézier curves that contain the co-called source point denoted as s_0 and the field point denoted as s, while $n_j^{(1)}(s)$, $n_j^{(2)}(s)$ are the normals to the curves.
- The declaration of Bézier curves in formula (2) is separated from the boundary functions $u_j(s)$ and $p_j(s)$ describing in the physical interpretation temperature and flux on the boundary. In this paper they are approximated by Chebyshev series

$$u_j(s) = \sum_{k=0}^{K-1} u_j^{(k)} T_j^{(k)}(s), \qquad p_j(s) = \sum_{k=0}^{K-1} p_j^{(k)} T_j^{(k)}(s), \tag{5}$$

  where $u_j^{(k)}$, $p_j^{(k)}$ denote the coefficients of the series, K is the adopted number of terms of the series, whereas $T_j^{(k)}(s)$ represent the Chebyshev polynomials defined in the parametric reference system in PIES.

- It should be noted, that the form of boundary functions (5) is completely independent from the Bézier curves used to describe the boundary. This is the essence of the formal separation of the boundary shape from the representation of the problem on the boundary in PIES. The direct incorporation of the boundary shape in functions (2) through kernels (3) is the main advantage of PIES compared to traditional BIE. As a result, the proposed PIES is characterized by the fact that its numerical solution does not require, in contrast to the BEM, discretization at the level of the aforementioned boundary declaration as well as the discretization of the boundary functions.
- Having the solutions on the boundary, in the second step we can obtain from them the solutions inside the domain using an additional integral identity represented by formula (10). There is no need to define a representation of the domain, so we have

ICCS Camera Ready Version 2021
To cite this paper please use the final published version:
DOI: 10.1007/978-3-030-77961-0_38

the possibility to find such solutions at any point and at the same time we can freely refine their resolution.

3 Elimination of singularities from PIES through the regularization

We want to eliminate the singularities from the integral functions $\overline{U}_{lj}^{*}(s_0,s)$, $\overline{P}_{lj}^{*}(s_0,s)$ through the regularization applied to (2). The presence of the singularities in (2) depends on the distance between the source s_0 and the field s points. If this distance is significant then the kernels $\overline{U}_{lj}^{*}(s_0,s)$ and $\overline{P}_{lj}^{*}(s_0,s)$ are easily numerically integrable with standard numerical methods, e.g. with the G-L quadrature rule. However, when this distance goes to zero $s \to s_0$, the integral functions become singular and the values of such integrals tend to infinity. We can identify weak (logarithmic) singularity in $\overline{U}_{lj}^{*}(s_0,s)$ and strong singularity in $\overline{P}_{lj}^{*}(s_0,s)$. The direct application of the G-L quadrature to evaluate the singular integrals produces large errors. In previous studies, we evaluated the strongly singular integrals analytically, whereas the weakly singular ones with singular points isolation. Finally, these integrals have been evaluated with satisfactory accuracy, but at the expense of additional complexity.

In order to regularize PIES represented by (2), we introduce below an auxiliary formula (2) but with other boundary functions marked as $\hat{u}_j(s)$, $\hat{p}_j(s)$

$$0.5\hat{u}_l(s_0)=\sum_{j=1}^{n}\int_0^1\left\{\overline{U}_{lj}^{\,*}(s_0,s)\hat{p}_j(s)-\overline{P}_{lj}^{\,*}(s_0,s)\hat{u}_j(s)\right\}J_j(s)ds, \qquad l=1,2,3..n. \tag{6}$$

Next, we assume that $\hat{u}_j(s)$ takes the following form

$$\hat{u}_j(s)=A_l(s_0)(\boldsymbol{\Gamma}_j(s)-\boldsymbol{\Gamma}_l(s_0))+u_l(s_0), \tag{7}$$

together with its directional derivative along to the normal vector to the boundary

$$\hat{p}_j(s)=A_l(s_0)n_j(s_0),\; n_j(s)=n_j^{(1)}(s)+n_j^{(2)}(s), \tag{8}$$

where $$A_l(s_0)=\frac{p_l(s_0)}{n_l(s_0)},\;\; n_l(s_0)=n_l^{(1)}(s_0)+n_l^{(2)}(s_0).$$

Function (7) is arbitrarily chosen to satisfy Laplace's equation. After subtracting (6) from (2), we get the final formula for the regularized PIES

DOI: 10.1007/978-3-030-77961-0_38

$$\sum_{j=1}^{n}\left\{\int_0^1 \overline{U}_{lj}^{\,*}(s_0,s)[p_j(s)-d_{lj}(s,s_0)p_l(s_0)]-\right.$$
$$\left.\int_0^1 \overline{P}_{lj}^{\,*}(s_0,s)[u_j(s)-u_l(s_0)-g_{lj}(s_0,s)p_l(s_0)]\right\}J_j(s)ds=0, \tag{9}$$

where $\quad d_{lj}(s,s_0)=\dfrac{n_j(s)}{n_l(s_0)}, \quad g_{lj}(s_0,s)=\dfrac{\boldsymbol{\Gamma}_j(s)-\boldsymbol{\Gamma}_l(s_0)}{n_l(s_0)}.$

After solving (9), we obtain solutions $u_j(s)$ and $p_j(s)$ on the boundary in the form of the Chebyshev series (5). Having these solutions on the boundary, in the second step we can obtain solutions inside the domain at point $\boldsymbol{x}\in\Omega$ using the following integral identity

$$u_\Omega(\boldsymbol{x})=\sum_{j=1}^{n}\int_0^1\left\{\overline{U}_j^{\,*}(\boldsymbol{x},s)p_j(s)-\overline{P}_j^{\,*}(\boldsymbol{x},s)u_j(s)\right\}J_j(s)ds, \quad \boldsymbol{x}\in\Omega. \tag{10}$$

Formula (10), similarly as (9), includes analytically the boundary generated by Bézier curves via following kernels

$$\overline{U}_j^{\,*}(\boldsymbol{x},s)=\frac{1}{2\pi}\ln\frac{1}{[\eta_1^2+\eta_2^2]^{0.5}}, \quad \overline{P}_j^{\,*}(\boldsymbol{x},s)=\frac{1}{2\pi}\frac{\eta_1 n_1^{(j)}(s)+\eta_2 n_2^{(j)}(s)}{\eta_1^2+\eta_2^2}, \tag{11}$$

where

$$\eta_1=x_1-\Gamma_j^{(1)}(s),\ \eta_2=x_2-\Gamma_j^{(2)}(s).$$

To determine the solution in the domain, only coefficients $u_j^{(k)}$ and $p_j^{(k)}$ for every Bézier curve which model the boundary have to be taken into account in (10).

4 Numerical implementation

In order to use (9) for simulating stationary temperature field, the collocation method [16] is applied. The collocation points are placed in the parametric domain of Bézier curves and represent by points s_0. Writing (9) at the collocation points, we obtain a system of algebraic equations approximating PIES with the size determined by the number of parametric curves modeling the boundary and the number of terms in the approximating series (5) on individual curves.

ICCS Camera Ready Version 2021
To cite this paper please use the final published version:
DOI: 10.1007/978-3-030-77961-0_38

$$
\begin{bmatrix}
\bar{h}_{11}^{(1,0)} & \cdots & h_{1j}^{(1,k)} & \cdots & h_{1n}^{(1,K-1)} \\
\vdots & & \vdots & & \vdots \\
h_{l1}^{(c,0)} & \cdots & \bar{h}_{lj}^{(c,k)} & \cdots & h_{ln}^{(c,K-1)} \\
\vdots & & \vdots & & \vdots \\
h_{n1}^{(K-1,0)} & \cdots & h_{nj}^{(K-1,k)} & \cdots & \bar{h}_{nn}^{(K-1,K-1)}
\end{bmatrix}
\begin{bmatrix}
p_1^{(0)} \\ \vdots \\ p_j^{(k)} \\ \vdots \\ p_n^{(K-1)}
\end{bmatrix}
=
\begin{bmatrix}
\bar{g}_{11}^{(1,0)} & \cdots & g_{1j}^{(1,k)} & \cdots & g_{1n}^{(1,K-1)} \\
\vdots & & \vdots & & \vdots \\
g_{l1}^{(c,0)} & \cdots & \bar{g}_{lj}^{(c,k)} & \cdots & g_{ln}^{(c,K-1)} \\
\vdots & & \vdots & & \vdots \\
g_{n1}^{(K-1,0)} & \cdots & g_{nj}^{(K-1,k)} & \cdots & \bar{g}_{nn}^{(K-1,K-1)}
\end{bmatrix}
\begin{bmatrix}
u_1^{(0)} \\ \vdots \\ u_j^{(k)} \\ \vdots \\ u_n^{(K)}
\end{bmatrix}. \tag{12}
$$

In the absence of regularization and direct application of formula (2) for the solution of the problem on the boundary, all integrals on the main diagonals of matrices **H** and **G** are singular. The proposed regularization eliminates these singularities and the new formulas for non-singular integrals on the main diagonal in (12) on the basis of (9) are as follows

$$
\begin{aligned}
[\bar{h}_{ll}^{(c,k)}] &= \int_0^1 \bar{U}^{*}{}_{ll}(s_0^{(l,c)},s)T_l^{(k)}(s)J_l(s)ds \\
&- \sum_{j=0}^{n}\int_0^1 \frac{n_j^{(1)}(s)+n_j^{(2)}(s)}{n_l^{(1)}(s_0^{(c)})+n_l^{(2)}(s_0^{(c)})}\bar{U}_{lj}{}^{*}(s_0^{(c)},s)T_j^{(k)}(s)J_j(s)ds \\
&+ \sum_{j=0}^{n}\int_0^1 \frac{\Gamma_j^{(1)}(s)-\Gamma_l^{(1)}(s_0^{(c)})+\Gamma_j^{(2)}(s)-\Gamma_l^{(2)}(s_0^{(c)})}{n_l^{(1)}(s_0^{(c)})+n_l^{(2)}(s_0^{(c)})}\bar{P}_{lj}{}^{*}(s_0^{(c)},s)T_j^{(k)}(s)J_j(s)ds,
\end{aligned} \tag{13}
$$

$$
[\bar{g}_{ll}^{(c,k)}] = \int_0^1 \bar{P}_{ll}{}^{*}(s_0^{(c)},s)T_l^{(k)}(s)J_l(s)ds - \sum_{j=0}^{n}\int_0^1 \bar{P}_{lj}{}^{*}(s_0^{(c)},s)T_j^{(k)}(s)J_j(s)ds. \tag{14}
$$

Non-diagonal elements in (12) are calculated on the basis of the following integrals

$$
[g_{lj}^{(c,k)}] = \int_0^1 \bar{P}_{lj}{}^{*}(s_0^{(c)},s)T_j^{(k)}(s)J_j(s)ds, \tag{15}
$$

$$
[h_{lj}^{(c,k)}] = \int_0^1 \bar{U}_{lj}{}^{*}(s_0^{(c)},s)T_j^{(k)}(s)J_j(s)ds. \tag{16}
$$

Integrals (15) and (16) are non-singular and have the same form both for (2) and (9). The complete algorithm for solving the regularized PIES is listed below.

Regularized PIES algorithm

Read boundary input data (control points of n Bézier segments)
Read boundary conditions
for $l \leftarrow 1, n$ **do** //loop over Bézier segments
  for $j \leftarrow 1, n$ **do** //loop over Bézier segments
    if $i == j$ **then**
      for $k \leftarrow 0, K-1$ **do** //loop over Chebyshev series
        for $c \leftarrow 1, K$ **do** //loop over collocation points

DOI: 10.1007/978-3-030-77961-0_38

for $e \leftarrow 1, n$ **do** //loop over Bézier segments

$[g_{ll}^{(kc)}] \leftarrow G\_L\_integration\left(\overline{P}^{*}{}_{ll}(s_0^{(c)}, s^{(q)}) T_l^{(k)}(s^{(q)}) J_l(s^{(q)}) - \overline{P}^{*}{}_{le}(s_0^{(c)}, s^{(q)}) T_e^{(k)}(s^{(q)}) J_e(s^{(q)})\right)$

$[h_{ll}^{(kc)}] \leftarrow G\_L\_integration\Big(\overline{U}^{*}{}_{ll}(s_0^{(c)}, s^{(q)}) T_l^{(k)}(s^{(q)}) J_l(s^{(q)})$
$-(n_e^{(1)}(s^{(q)}) + n_e^{(2)}(s^{(q)}))/(n_l^{(1)}(s_0^{(c)}) + n_l^{(2)}(s_0^{(c)}))\overline{U}^{*}{}_{le}(s_0^{(c)}, s^{(q)}) T_e^{(k)}(s^{(q)}) J_e(s^{(q)})$
$-(\Gamma_e^{(1)}(s^{(q)}) - \Gamma_l^{(1)}(s_0^{(c)}) + \Gamma_e^{(2)}(s^{(q)}) - \Gamma_l^{(2)}(s_0^{(c)}))/(n_l^{(1)}(s_0^{(c)}) + n_l^{(2)}(s_0^{(c)}))\overline{P}^{*}{}_{le}(s_0^{(c)}, s^{(q)}) T_e^{(k)}(s^{(q)}) J_e(s^{(q)})\Big)$

end for
end for
end for
add submatrix $[g_{ll}^{(kc)}]$ to $[g_{ll}]$ and $[h_{ll}^{(kc)}]$ to $[h_{ll}]$
else
for $k \leftarrow 0, K-1$ **do** //loop over Chebyshev series
for $c \leftarrow 1, K$ **do** //loop over collocation points
$[g_{lj}^{(kc)}] \leftarrow G\_L\_integration\left(\overline{P}^{*}{}_{lj}(s_0^{(c)}, s^{(q)}) T_j^{(k)}(s^{(q)}) J_j(s^{(q)})\right)$
$[h_{lj}^{(kc)}] \leftarrow G\_L\_integration\left(\overline{U}^{*}{}_{lj}(s_0^{(c)}, s^{(q)}) T_j^{(k)}(s^{(q)}) J_j(s^{(q)})\right)$
end for
end for
add submatrix $[g_{lj}^{(kc)}]$ to $[g_{lj}]$ and $[h_{lj}^{(kc)}]$ to $[h_{lj}]$
end if
add submatrix $[g_{lj}]$ to $\mathbf{G}$ and $[h_{lj}]$ to $\mathbf{H}$
end for
end for
applying boundary conditions
transform $[\mathbf{H}]\{\mathbf{u}\} = [\mathbf{G}]\{\mathbf{p}\}$ into $[\mathbf{A}]\{\mathbf{x}\} = \{\mathbf{b}\}$
solve system of equations $[\mathbf{A}]\{\mathbf{x}\} = \{\mathbf{b}\}$

---

5 Verification of the approach

The regularization is validated on two examples having analytical solutions. Below, we show how to generate the boundary by Bézier curves and investigate the influence of the minimal distance between the collocation point and the quadrature node on the stability of diagonal integrals and overall accuracy of PIES.

5.1 Example 1

We consider a stationary temperature distribution governed by the Laplace equation in a wrench. As shown in Fig. 2a, the boundary is generated by 14 Bézier curves. Among them, 9 are linear being simply straight lines between two end points. The remaining 5

ICCS Camera Ready Version 2021
To cite this paper please use the final published version:
DOI: 10.1007/978-3-030-77961-0_38

are the cubic ones each defined by 4 control points and used to define curvilinear parts of the boundary.

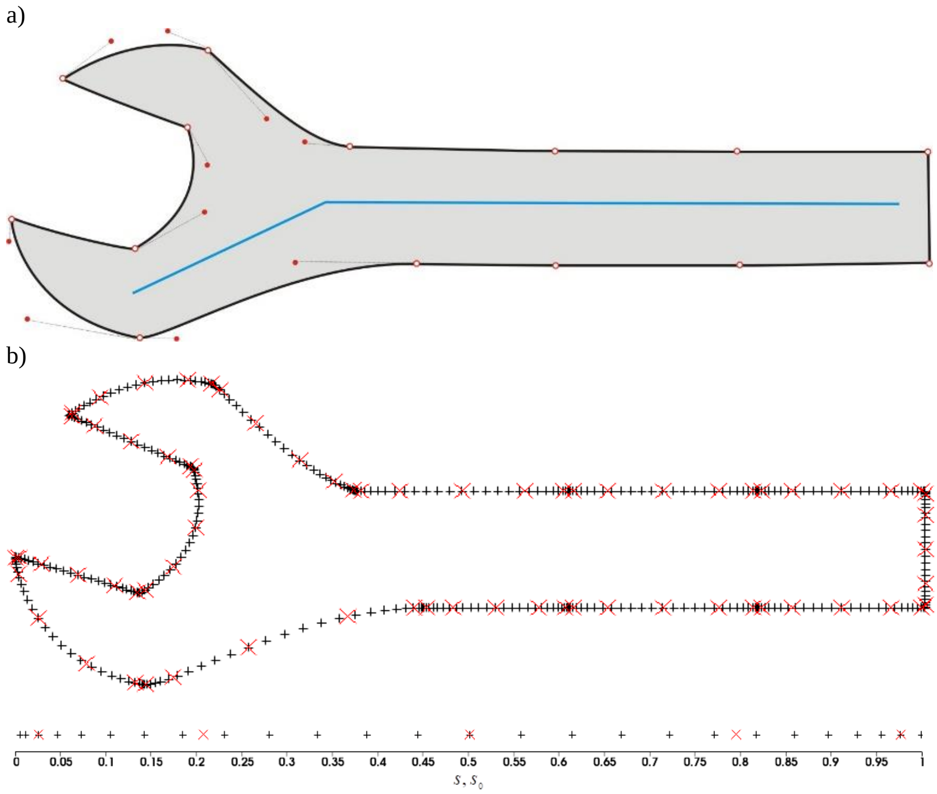

Fig. 2. The boundary of the problem for example 1 defined by Bézier curves together with the analyzed cross-section of the domain solutions (a), distribution of 5 collocation points (red x) and 27 nodes of the G-L quadrature (black +) in the parametric reference system and after mapping to the boundary (b).

We assume that the expected distribution of temperature on the boundary and inside the domain is described by the following function that depends on the Cartesian coordinates

$$u(x_1,x_2) = -x_1^3 - -x_2^3 + 3x_1^2 x_2 + 3x_1 x_2^2. \tag{17}$$

The value of function (17) is specified on the boundary in the form of Dirichlet boundary conditions, while its normal derivative

$$\frac{du(x_1,x_2)}{dn} = (-3x_1^2 + 6x_1x_2 + 3x_2^2)n_1 + (-3x_2^2 + 6x_1x_2 + 3x_1^2)n_2\,, \tag{18}$$

represents the expected analytical solutions on the boundary.

ICCS Camera Ready Version 2021
To cite this paper please use the final published version:
DOI: 10.1007/978-3-030-77961-0_38

In order to solve the problem on the boundary by (9), we place 5 collocation points at roots of the Chebyshev polynomials of the second kind within the parametric domain of each Bézier cuve. The nodes of the G-L quadrature of degree 27 are defined in the same parametric domain. Due to full parameterization of the boundary, we can freely choose the positions of the collocation points and quadrature nodes identified with s_0 and s in (9) and also in (2). Their mutual distribution in the parametric domain reference system s, s_0 and after mapping to each of the 14 Bézier curves is shown in Fig. 2b. It should be noted that formula (2) is singular for every collocation point. The proposed regularization eliminates this problem.

Below, we examine how the distance between collocation points and quadrature nodes influences the stability of the integrals on the diagonal for formula (2) and the regularized one (9). Fig. 2b indicates the coverage for the central collocation point $s_0 = 0.5$ exactly with the central quadrature node for each Bézier curve. We decide to move this collocation point to study the influence of the minimum distance between s_0 and s on stability of these integrals. Fig. 3 presents a summary of this analysis.

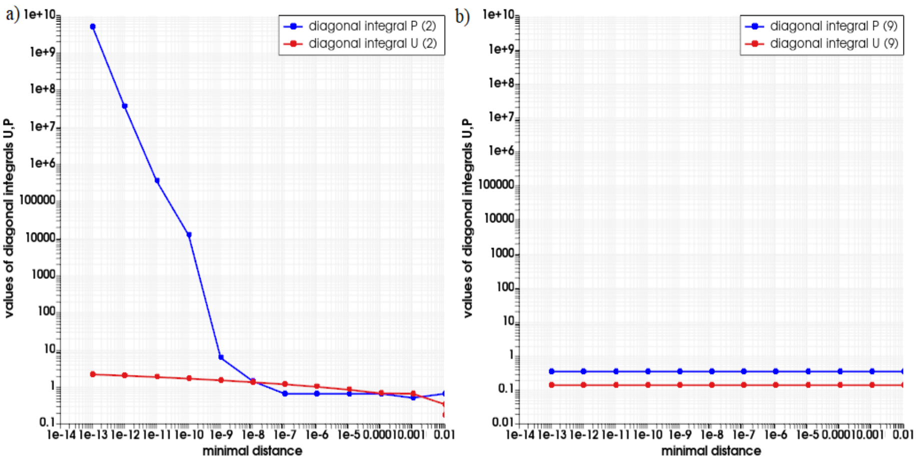

Fig. 3. The influence of the minimal distance between the collocation point and the quadrature node on the stability of diagonal integrals for (2) (a) and (9) (b).

Fig. 3a shows that the diagonal integrals in (2) with the direct application of G-L quadrature are unstable. It is especially noticeable for the strongly singular one. In turn, Fig. 3b shows that the diagonal integrals in (9) are stable for the full range of distances between the investigated collocation point and quadrature node. The presented results refer to one selected collocation point from the total number of 70 specified in Fig. 2b. Moreover, these behaviors and dependencies are analogous for all other points. It should be noted that the values of diagonal integrals in (2) and (9) are different due to the regularization. But at this point we are interested in forecasting overall computational stability rather than individual values.

The regularization also allows for obtaining excellent accuracy of the problem under study. Fig. 4 shows the solution on the boundary obtained by (9) and in the domain by

ICCS Camera Ready Version 2021
To cite this paper please use the final published version:
DOI: 10.1007/978-3-030-77961-0_38

identity (10) for the case when the minimal distance between the collocation point and the quadrature node is 1e-13.

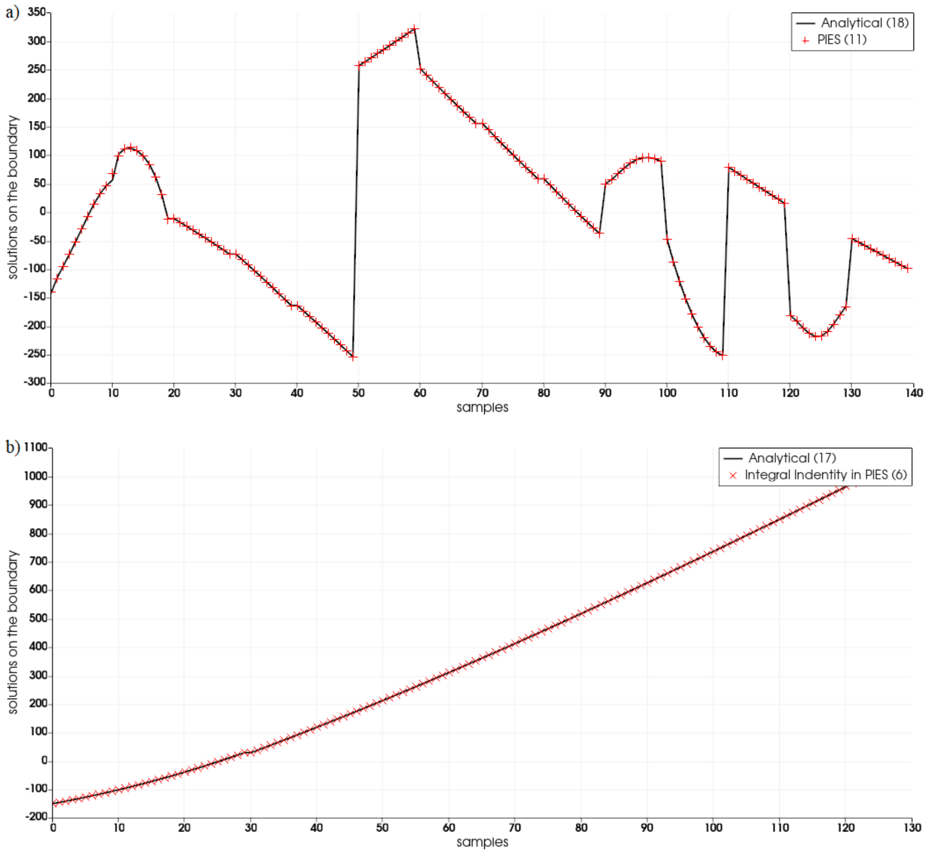

Fig. 4. The obtained solutions with the regularization on the boundary (a) and in the domain (b) when the minimal distance between the collocation point and the quadrature node is 1e-13.

The results show excellent agreement with exact solutions (17-18) and confirm the strategy, which is independent from the representation of the boundary shape and the type of applied boundary conditions.

5.2 Example 2

We repeat the analysis given in example 1, but for more complicated shape of the boundary with another boundary conditions. We consider a stationary temperature distribution in a multiply connected domain shown in Fig. 5a. The inner and outer boundaries are described by linear Bézier curves. The geometry of the boundary is thus completely defined by a set of 31 control points.

ICCS Camera Ready Version 2021
To cite this paper please use the final published version:
DOI: 10.1007/978-3-030-77961-0_38

a)

b)

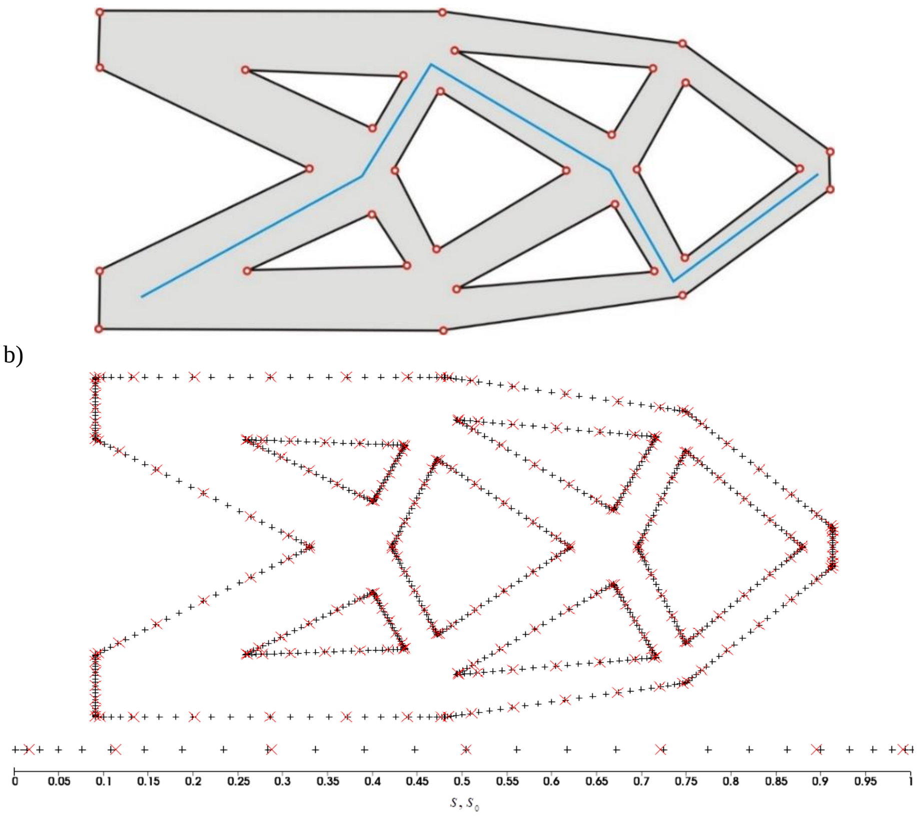

Fig. 5. The boundary of the problem for example 2 defined by Bézier curves together with the analyzed cross-section of the domain solutions (a), distribution of 7 collocation points (red x) and 27 nodes in the G-L quadrature (black +) in the parametric reference system and after mapping to the boundary (b).

We assume that Dirichlet boundary conditions are posed on the whole boundary. They are calculated on the basis of the following function

$$u(x_1, x_2) = \cos(x_1)\exp(x_2). \tag{19}$$

Function (19) satisfies the Laplace equation and represents the expected analytical temperature distribution inside the multiply connected domain. In turn, the normal derivative of (19) gives the reference analytical solutions of the problem on the boundary

$$\frac{du(x_1, x_2)}{dn} = -\sin(x_1)\exp(x_2)n_1 + \cos(x_1)\exp(x_2)n_2. \tag{20}$$

Table 1 shows the influence of the minimal distance between collocation point and the quadrature node for accuracy of solutions on the boundary obtained by (9) and in the

ICCS Camera Ready Version 2021
To cite this paper please use the final published version:
DOI: 10.1007/978-3-030-77961-0_38

domain by identity (10). The results are computed for 7 collocation points and 27 nodes of the G-L quadrature per Bézier curve. When analyzing the placement of these points and nodes shown in Fig. 5b, we can identify the cases with $s \to s_0$ and $s_0 = s$. Moreover, as in example 1, the central collocation point coincides with the central quadrature node. Therefore, we again decide to move this point to determine the minimum distance, for which we observe the existence and stability of the solutions.

Table 1. The influence of the minimal distance between collocation point and G-L quadrature nod for accuracy of solutions on the boundary and domain.

Minimal distance between collocation and quadrature points	L_2 error norm [%] for solutions on the boundary	L_2 error norm [%] for solutions in the domain
1e-2	0.396046	0.00730305
1e-3	0.397675	0.00739621
1e-4	0.398794	0.00741943
1e-5	0.398975	0.00742293
1e-6	0.398999	0.00742338
1e-7	0.399002	0.00742344
1e-8	0.399002	0.00742345
1e-9	0.399002	0.00742345
1e-10	0.399002	0.00742419
1e-11	0.398997	0.00742788
1e-12	0. 398991	0.00742783
1e-13	0.400376	0.00770156
0.0	0.399002	0.00742345

The results again confirm the stability of solutions for the multi connected domain. We obtained the excellent accuracy of the regularized PIES for a very close distance between collocation points and quadrature nodes.

6 Conclusions

The results indicate the effectiveness of the proposed regularization. It avoids the use of complicated explicit methods for the evaluation of singular integrals and, on the other hand, provides a unified scheme for eliminating these singularities. The approach is also independent from the ways of declaring the boundary with the help of various curves that have already been used in PIES. In the paper, Bézier curves are chosen, but we can apply other ones, e.g. NURBS. Moreover, the separation of the boundary declaration from the approximation of the boundary functions, as in the original PIES's formula is preserved. Thanks to this, in the current paper the boundary functions could be approximated with an effective Chebyshev series.

ICCS Camera Ready Version 2021
To cite this paper please use the final published version:
DOI: 10.1007/978-3-030-77961-0_38

References

1. Zieniuk, E.: Bézier curves in the modification of boundary integral equations (BIE) for potential boundary-values problems. International Journal of Solids and Structures 40(9), 2301-2320 (2003).
2. Zieniuk, E., Szerszeń, K.: The PIES for solving 3D potential problems with domains bounded by rectangular Bézier patches. Engineering Computations 31(4), 791-809 (2014).
3. Zieniuk, E., Szerszeń, K: A separation of the boundary geometry from the boundary functions in PIES for 3D problems modeled by the Navier–Lamé equation. Computers & Mathematics with Applications 75(4), 1067-1094 (2018).
4. Telles, J.C.F.: A self-adaptive co-ordinate transformation for efficient numerical evaluation of general boundary element integrals. International Journal for Numerical Methods in Engineering. 24(5), 959-973 (1987).
5. Cerrolaza, M., Alarcon, E.: A bi-cubic transformation for the numerical evaluation of the Cauchy principal value integrals in boundary methods. International Journal for Numerical Methods in Engineering 28(5), 987-999 (1989).
6. Li, X., Su, Y.: Three-dimensional stress analysis of thin structures using a boundary element method with sinh transformation for nearly singular integrals. Computers & Mathematics with Applications. 72(11), 2773-2787 (2016).
7. Gao, X.W., Yang, K., Wang, J.: An adaptive element subdivision technique for evaluation of various 2D singular boundary integrals. Engineering Analysis with Boundary Elements 32(8), 692-696 (2008).
8. Xie, G., Zhang, J., Qin, X., Li, G.: New variable transformations for evaluating nearly singular integrals in 2D boundary element method. Engineering Analysis with Boundary Elements, 35(6), 811-817 (2011).
9. Fata, S.N.: Semi-analytic treatment of nearly-singular Galerkin surface integrals. Applied Numerical Mathematics 60(10), 974-993 (2010).
10. Niu, Z., Wendland, W.L., Wang, X., Zhou, H.: A semi-analytical algorithm for the evaluation of the nearly singular integrals in three-dimensional boundary element methods. Computer Methods in Applied Mechanics and Engineering 194(9-11), 1057-1074 (2005).
11. Hayami, K., Matsumoto, H.: A numerical quadrature for nearly singular boundary element integrals. Engineering Analysis with Boundary Elements 13(2), 143-154 (1994).
12. Chen, H.B., Lu, P., Schnack, E.: Regularized algorithms for the calculation of values on and near boundaries in 2D elastic BEM. Engineering Analysis with Boundary Elements 25(10), 851-876 (2001).
13. Granados, J.J., Gallego, R.: Regularization of nearly hypersingular integrals in the boundary element method. Engineering Analysis with Boundary Elements 25(3), 165-184 (2001).
14. Sladek, V., Sladek, J., Tanaka, M.: Regularization of hypersingular and nearly singular integrals in the potential theory and elasticity. International Journal for Numerical Methods in Engineering 36(10), 1609-1628 (1993).
15. Huang, Q., Cruse, T.A.: Some notes on singular integral techniques in boundary element analysis. International Journal for Numerical Methods in Engineering 36(15), 2643-2659 (1993).
16. Gottlieb, D., Orszag, S.A.: Numerical analysis of spectral methods: theory and applications. SIAM, Philadelphia (1977).

ICCS Camera Ready Version 2021
To cite this paper please use the final published version:
DOI: 10.1007/978-3-030-77961-0_38