Analytic and Numerical Solutions of Space-Time Fractional Diffusion Wave Equations with different Fractional order*

Abhishek Kumar Singh[1 [0000−0002−1263−687X]] and Mani Mehra[1 [0000−0002−4261−7784]]

Indian Institute of Technology Delhi, New Delhi-110016, India
assinghabhi@gmail.com
mmehra@maths.iitd.ac.in

Abstract. The aim of this manuscript is to investigate analytic and numerical solutions of space–time fractional diffusion wave equations with different fractional order (α and β). After deriving analytic solution, an implicit unconditional stable finite difference method for solving space-time fractional diffusion wave equations is proposed. The Gerschgorin theorem is used to study the stability and convergence of the method. Furthermore, the behavior of the error is examined to verify the order of convergence by numerical example.

Keywords: Implicit finite difference method · Riesz space-fractional derivative · Stability · Convergence.

1 Introduction

In fluid mechanics [10], physics [3], biology [4], system control [13], hydrology [12], finance [15, 11, 14, 16] and various other engineering fields [7], we often encounter complex systems which cannot be modeled by the conventional integer order differential equations. In such scenarios, the use and study of differential equations with non-integer powers of the differentiation order, namely the "fractional differential equations (FDEs)" proves to be extremely beneficial. FDEs describe the memory and hereditary properties of different substances which the conventional models are incapable to take care of.

Several numerical methods have been used to obtained approximate solutions to FDEs including finite difference method, finite element method and the spectral method. The finite difference method is the most efficient numerical method and is a powerful tool owing to its easy implementation. It is used for solving the time and/or space fractional diffusion wave equations. For instance, Khader *et. al.* [6] obtained the numerical solutions of time fractional diffusion wave equations by using the Hermite formula. Sun and Wu [17] and Sweilam *et al.* [18] obtained the numerical solution of time and two sided space fractional wave equations, respectively, using finite difference scheme. Chen and Li

* Supported by University Grants Commission, New Delhi-110002, India.

ICCS Camera Ready Version 2021
To cite this paper please use the final published version:
DOI: 10.1007/978-3-030-77961-0_34

[1] and Zhang *et. al.* [19] used a compact finite difference scheme to obtain the approximate solution for a time fractional diffusion wave equations.

This paper is concerned with the space-time fractional diffusion wave equation. We derived the finite difference approximation for the following initial boundary value problem:

$$\begin{cases} {}_{\mathrm{C}}D^{\alpha}_{0,t}u(x,t) = -(-\Delta)^{\beta/2}u(x,t) + g(x,t), \quad 0 < t < T, 0 < x < 1, \\ u(0,t) = u(1,t) = 0, \quad 0 < t < T, \\ u(x,0) = v_1(x), \ \frac{\partial u(x,0)}{\partial t} = v_2(x), \quad 0 < x < 1, \end{cases} \tag{1}$$

where $1 < \alpha < 2$, $1 < \beta < 2$, ${}_{\mathrm{C}}D^{\alpha}_{0,t}$ denote the left-side Caputo fractional derivative of order α with respect to t and $-(-\Delta)^{\beta/2}$ is the Riesz space-fractional derivative of order β.

The article is drafted as follows. Section 2 provides some basic definitions and theorems along with the derivation of the analytic solution of the proposed problem. Section 3 describes the formulation of the implicit finite difference scheme. In section 4, we present the unconditional stability of the implicit finite difference scheme. We also investigate the convergence and error estimate of the scheme. In section 5, a numerical example with known exact solution is presented to verify the convergence.

2 Preliminaries

In this Section, we present some basic concepts required for our work. These have been taken from [9, 2, 8].

Definition 1. *The left and right Riemann-Liouville derivatives with $\beta > 0$ of the given function $f(t)$, $t \in (a,b)$ are defined as*

$$_{RL}D^{\beta}_{a,t}f(t) = \frac{1}{\Gamma(m-\beta)}\frac{d^m}{dt^m}\int_a^t (t-s)^{m-\beta-1}f(s)ds,$$

and

$$_{RL}D^{\beta}_{t,b}f(t) = \frac{(-1)^m}{\Gamma(m-\beta)}\frac{d^m}{dt^m}\int_t^b (s-t)^{m-\beta-1}f(s)ds,$$

respectively, where m is a positive integer satisfying $m-1 \leq \beta < m$.

Definition 2. *The left and right $\text{Grünwald} - \text{Letnikov}$ derivatives with order $\beta > 0$ of the given function $f(t)$, $t \in (a,b)$ are defined as*

$$_{GL}D^{\beta}_{a,t}f(t) = \lim_{\substack{h\to 0\\ Nh=t-a}} h^{-\beta}\sum_{j=0}^{N}(-1)^j\binom{\beta}{j}f(t-jh),$$

and

$$_{GL}D^{\beta}_{t,b}f(t) = \lim_{\substack{h\to 0\\ Nh=b-t}} h^{-\beta}\sum_{j=0}^{N}(-1)^j\binom{\beta}{j}f(t+jh).$$

ICCS Camera Ready Version 2021
To cite this paper please use the final published version:
DOI: 10.1007/978-3-030-77961-0_34

Definition 3. *The left Caputo derivative with order $\alpha > 0$ of the given function $f(t),\ t \in (a, b)$ is defined as*

$$_{C}D^{\alpha}_{a,t}f(t) = \frac{1}{\Gamma(m-\alpha)} \int_a^t (t-s)^{m-\alpha-1} f^m(s)ds,$$

where m is a positive integer satisfying $m - 1 < \alpha \leq m$.

Definition 4. *A real or complex-valued function $f(x)$, $x > 0$, is said to be in the space C_γ, $\gamma \in \mathbb{R}$, if there exists a real number $p > \gamma$ such that*

$$f(x) = x^p f_1(x),$$

for a function $f_1(x)$ in $C([0, \infty])$.

Definition 5. *A function $f(x)$, $x > 0$, is said to be in the space C^m_γ, $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, if and only if $f^m \in C_\gamma$.*

Definition 6 (Multivariate Mittag-Leffler fuction). *A multivariate Mittag-Leffler function $E_{(a_1,...,a_n),b}(z_1, ..., z_n)$ of n complex variables $z_1, ..., z_n \in \mathbb{C}$ with parameters $a_1, ..., a_n, b$, is defined as*

$$E_{(a_1,...,a_n),b}(z_1, ..., z_n) = \sum_{k=0}^{\infty} \sum_{\substack{l_1+...+l_n=k \\ l_1,...,l_n \geq 0}} (k; l_1, ..., l_n) \frac{\prod_{i=1}^n z_i^{l_i}}{\Gamma(b + \sum_{i=1}^n a_i l_i)},$$

in terms of multinomial coefficients

$$(k; l_1, ..., l_n) = \frac{k!}{l_1!...l_n!}, \quad k, l_1, ..., l_n \in \mathbb{N},$$

where $b > 0$, and all $a_i > 0$. In particular, if $n = 1$, the multivariate Mittag-Leffler function is reduced to the Mittag-Leffler function

$$E_{a_1,b}(z_1) = \sum_{k=0}^{\infty} \frac{z_1^k}{\Gamma(b + ka_1)}, \quad a_1 > 0,\ b > 0,\ z_1 \in \mathbb{C}.$$

The Mittag-Leffler function $E_{a_1,b}(z_1)$ is a two parameter family of entire functions in z_1 of order a_1^{-1}. It generalizes the exponential function in the sense that $E_{1,1}(z_1) = e^{z_1}$.

Definition 7. *The Riesz derivative with order $\beta > 0$ of the given function $f(x),\ x \in (a, b)$ is defined as*

$$_{RZ}D^{\beta}_{x}f(x) = c_\beta(_{RL}D^{\beta}_{a,x}f(x) + {}_{RL}D^{\beta}_{x,b}f(x)),$$

where $c_\beta = -\frac{1}{2\cos(\beta\pi/2)}$, $\beta \neq 2k + 1, k = 0, 1, \cdots$. $_{RZ}D^{\beta}_{x}f(x)$ is sometimes expressed as $\frac{d^\beta}{d|x|^\beta}$ or $-(-\Delta)^{\beta/2}$.

ICCS Camera Ready Version 2021
To cite this paper please use the final published version:
DOI: 10.1007/978-3-030-77961-0_34

Using definition (7) in equation (1), we get

$$\begin{cases} {}_{\mathrm{C}}D^{\alpha}_{0,t}u(x,t) = c_{\beta}({}_{\mathrm{RL}}D^{\beta}_{0,x}u(x,t) + {}_{\mathrm{RL}}D^{\beta}_{x,1}u(x,t)) + g(x,t), \quad 0 < t < T, 0 < x < 1, \\ u(0,t) = u(1,t) = 0, \quad 0 < t < T, \\ u(x,0) = v_1(x), \ \frac{\partial u(x,0)}{\partial t} = v_2(x), \quad 0 < x < 1, \end{cases} \tag{2}$$

where $T > 0$ is a fixed time, ${}_{\mathrm{RL}}D^{\beta}_{0,x}$ is the left Riemann-Liouville derivative and ${}_{\mathrm{RL}}D^{\beta}_{x,1}$ is the right Riemann-Liouville derivative.

Theorem 1. *Let* $\mu > \mu_1 > \cdots > \mu_n \geq 0$, $m_i - 1 < \mu_i \leq m_i$, $m_i \in \mathbb{N}_0$, $\lambda_i \in \mathbb{R}$, $i = 1, \cdots, n$. *The initial value problem*

$$\begin{cases} ({}_{C}D^{\mu}_{0,t}y)(t) - \sum_{i=1}^{n} \lambda_i ({}_{C}D^{\mu_i}_{0,t}y)(t) = g(t), \\ y^{(k)}(0) = c_k \in \mathbb{R}, \ k = 0, \cdots, m-1, \ m-1 < \mu \leq m, \end{cases} \tag{3}$$

where g is assumed to lie in C_{-1} if $\mu \in \mathbb{N}$ or in C^1_{-1} if $\mu \notin \mathbb{N}$, then (3) has a unique solution in the space C^m_{-1} of the form

$$y(t) = y_g(t) + \sum_{k=0}^{m-1} c_k u_k(t),$$

where

$$y_g(t) = \int_0^t s^{\mu-1} E_{(.),\mu}(s) g(t-s) ds,$$

and

$$u_k(t) = \frac{t^k}{k!} + \sum_{i=l_k+1}^{n} \lambda_i t^{k+\mu-\mu_i} E_{(.),k+1+\mu-\mu_i}(x), k = 0, ..., m-1,$$

fulfills the initial conditions $u^l_k(0) = \delta_{kl}$, $k, l = 0, ..., m-1$. Here,

$$E_{(.),\beta}(t) = E_{\mu-\mu_1,...,\mu-\mu_1,\beta}(\lambda_1 t^{\mu-\mu_1}, ..., \lambda_n t^{\mu-\mu_n}).$$

Proof. See [9].

Remark 21 *In Theorem 1, the natural numbers l_k, $k = 0, ..., m-1$, are determined from the condition $m_{l_k} \geq k+1$ and $m_{l_k+1} \leq k$. In the case $m_i \leq k$, $i = 0, \cdots, m-1$, we set $l_k = 0$ and if $m_i \geq k+1, i = 0, \cdots, m-1$ then $l_k = n$.*

Theorem 2. *Let H be a Hilbert space. If $\{\lambda_n\}_{n\geq 1}$ and $\{\phi_n\}_{n\geq 1}$ are the eigenvalues and eigenvectors associated to an operator A in H, then $\{\lambda^{\alpha}_n\}_{n\geq 1}$ and $\{\phi_n\}_{n\geq 1}$ are the eigenvalues and eigenvectors to the fractional operator A^{α}, $-1 < \alpha \leq 1$.*

Proof. See [2]

ICCS Camera Ready Version 2021
To cite this paper please use the final published version:
DOI: 10.1007/978-3-030-77961-0_34

Now we derive one lemma required for the present study using the aforementioned theorems.

Lemma 1. *The solution u of the problem (1) with* $1 < \alpha < 2$ *and* $1 < \beta \leq 2$ *is given by*

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t)\phi_n(x) = \sum_{n=1}^{\infty} \Big[\int_0^t s^{\alpha-1} E_{\alpha,\alpha}(-a_n s^{\alpha}) g_n(t-s)ds + \sum_{k=0}^{1} c_{k,n} y_k(t) \Big] \phi_n(x), \tag{4}$$

Proof : In order to solve the non-homogeneous equation (1), first we solve the corresponding homogeneous equation (by replacing $g(x,t) = 0$). We first Substitute $u(x,t) = X(x)T(t)$ in the homogeneous equation and obtain a fractional differential equation in $X(x)$:

$$\begin{cases} (-\Delta)^{\beta/2} X(x) = aX(x), \quad 0 < x < 1, \\ X(0) = 0 = X(1), \end{cases} \tag{5}$$

and a fractional linear differential equation with the Caputo derivative in $T(t)$

$${}_{\mathrm{C}}D_{0,t}^{\alpha} T(t) + aT(t) = 0, \tag{6}$$

where the parameter a is a positive constant.
Applying theorem (2) with $A = -\Delta$, $\alpha = \beta/2$, $H = L^2(0,1)$ and $D(A) = H^2(0,1) \cap H_0^1(0,1)$ there exists $\{\lambda_n^{\beta/2}\}_{n\geq 1}$, $\{\phi_n\}_{n\geq 1}$ eigenvalues and eigenvector of problem (5). Note that for $a = a_n$, $n \geq 1$ we have $X = X_n = \phi_n$ and $a_n = \lambda_n^{\beta/2}$. We now seek a solution of (1) of the form

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t)\phi_n(x), \tag{7}$$

where we assume that the series can be differentiated term by term. In order to determine $u_n(t)$, we expand $g(x,t)$ in the orthonormal complete system $\{\phi_n\}_{n\geq 1}$

$$g(x,t) = \sum_{n=1}^{\infty} g_n(t)\phi_n(x), \tag{8}$$

where

$$g_n(t) = \int_0^1 g(x,t)\phi_n(x)dx.$$

Substituting (7), (8) into (1) yields

$$\sum_{n=1}^{\infty} \phi_n(x)[{}_{\mathrm{C}}D_{0,t}^{\alpha} u_n(t)] = \sum_{n=1}^{\infty} \phi_n(x)[-a_n u_n(t) + g_n(t)], \tag{9}$$

ICCS Camera Ready Version 2021
To cite this paper please use the final published version:
DOI: 10.1007/978-3-030-77961-0_34

where we have used the fact that $\phi_n(x)$ is a solution of (5). Because $\{\phi_n\}_{n\geq 1}$ is an orthonormal system, multiplying both members of (9) by ϕ_n and integrate over (0,1) we get

$$ {}_{\mathrm{C}}D^{\alpha}_{0,t}u_n(t) + a_n u_n(t) = g_n(t). \tag{10}$$

On the other hand, because $u(x,t)$ satisfies the initial condition in (1), we must have

$$\sum_{n=1}^{\infty} \partial_t^{(k)} u_n(0)\phi_n(x) = v_k(x), \quad k = 0,1, \ x \in [0,1],$$

which implies

$$\partial_t^{(k)} u_n(0) = \int_0^1 v_k(x)\phi_n(x)dx := c_{k,n}, \quad n \geq 1, \ k = 0,1. \tag{11}$$

Finally, for each value of n, (10) and (11) constitute a fractional initial value problem. According to theorem (1), the fractional initial value problems (10) and (11) has the analytic solution

$$u_n(t) = \int_0^t s^{\alpha-1}E_{\alpha,\alpha}(-a_n s^{\alpha} g_n(t-s)ds + \sum_{k=0}^{1} c_{k,n}y_k(t), \quad t \geq 0, \tag{12}$$

where

$$y_k(t) = \frac{t^k}{k!} + \sum_{i=l_k+1}^{1} \xi_i t^{k+\alpha} E_{\alpha,k+1+\alpha}(\xi_i t^{\alpha}), \quad k = 0,1.$$

According to remark (21) $l_k = 0$, then

$$y_k(t) = \frac{t^k}{k!} + \xi_1 t^{k+\alpha} E_{\alpha,k+1+\alpha}(\xi_1 t^{\alpha}), \quad k = 0,1$$

with $\xi_1 = -a_n$. Hence we get the analytic solution of the initial boundary value problem (1)

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t)\phi_n(x) = \sum_{n=1}^{\infty} \big[\int_0^t s^{\alpha-1}E_{\alpha,\alpha}(-a_n s^{\alpha})g_n(t-s)ds + \sum_{k=0}^{1} c_{k,n}y_k(t)\big]\phi_n(x).$$

□

3 Finite Difference Approximation

To establish the numerical approximation scheme, we define the following notations. Let $\tau = T/K$ be the grid size in time direction with $t_n = n\tau$ $(n = 1, \cdots, K)$ and $h = 1/N$ be the grid size in spatial direction with $x_i = ih$ $(i = 0, 1, \cdots, N)$. Also, let $u_i^n \approx u(x_i, t_n)$, and $g_i^n = g(x_i, t_n)$. The finite difference approximation for Caputo fractional derivative appeared in problem (2) is derived as follows [8]

$$ {}_{\mathrm{C}}D^{\alpha}_{0,t}u(x_i, t_{n+1}) = \frac{1}{\Gamma(2-\alpha)} \sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} s^{1-\alpha}u''(x_i, t_n - s)ds.$$

ICCS Camera Ready Version 2021
To cite this paper please use the final published version:
DOI: 10.1007/978-3-030-77961-0_34

On each subinterval $[t_j, t_{j+1}]$, $u''(x_i, t_{n+1}-s)$ is approximated by $\frac{u(x_i,t_{n-j-1})-2u(x_i,t_{n-j})+u(x_i,t_{n-j+1})}{\tau^2}$, then the derived L2 scheme is

$$ {}_{\mathrm{C}}D_{0,t}^{\alpha}u(x_i,t_{n+1}) \approx \frac{\tau^{-\alpha}}{\Gamma(3-\alpha)}\sum_{j=0}^{n} b_j[u(x_i,t_{n-j-1}) - 2u(x_i,t_{n-j}) + u(x_i,t_{n+1-j})], \tag{13} $$

where $b_j = (j+1)^{2-\alpha} - j^{2-\alpha}$. For the approximation of left and right Riemann-Liouville fractional order derivative, we use the left and right shifted *Grünwald–Letnikov* formula (one shift) defined as [8]

$$ {}_{\mathrm{RL}}D_{0,x}^{\beta}u(x_i,t_n) \approx \frac{1}{h^{\beta}}\sum_{j=0}^{i+1} w_j^{(\beta)}u(x_{i-j+1},t_n), \tag{14} $$

and

$$ {}_{\mathrm{RL}}D_{x,1}^{\beta}u(x_i,t_n) \approx \frac{1}{h^{\beta}}\sum_{j=0}^{N-i+1} w_j^{(\beta)}u(x_{i+j-1},t_n), \tag{15} $$

respectively.

Let $\lambda = \frac{\tau^{\alpha}\Gamma(3-\alpha)}{h^{\beta}2cos(\beta\pi/2)}$, and $\lambda' = \tau^{\alpha}\Gamma(3-\alpha)$ then using (13), (14) and (15) at node point (x_i, t_{n+1}) in (2) we have the following implicit difference scheme

$$ \begin{aligned} \lambda\Big[\sum_{j=0}^{i+1} w_j^{(\beta)}u_{i-j+1}^{n+1} + \sum_{j=0}^{N-i+1} w_j^{(\beta)}u_{i+j-1}^{n+1}\Big] + u_i^{n+1} = \lambda' g_i^{n+1} - \sum_{j=1}^{n} b_j(u_i^{n-j-1} \\ -2u_i^{n-j} + u_i^{n-j+1}) - u_i^{n-1} + 2u_i^{n} \end{aligned} \tag{16} $$

We know that $\frac{\partial u(x,0)}{\partial t} = v_2(x)$, therefore

$$ u_i^{-1} = u_i^{1} - 2\tau v_2(x_i), \quad i = 0, 1, \cdots, N, $$

hence for $n = 0$ we have

$$ \lambda\Big[\sum_{j=0,j\neq1}^{i+1} w_j^{(\beta)}u_{i-j+1}^{1} + \sum_{j=0,j\neq1}^{N-i+1} w_j^{(\beta)}u_{i+j-1}^{1}\Big] + 2(1+\lambda w_1^{(\beta)})u_i^{1} = \lambda' g_i^{1} + 2\tau v_2(x_i) + 2v_1(x_i), \tag{17} $$

for $n \neq 0$, equation (16) is rewritten as

$$ \begin{aligned} \lambda\Big[\sum_{j=0,j\neq1}^{i+1} w_j^{(\beta)}u_{i-j+1}^{n+1} + \sum_{j=0,j\neq1}^{N-i+1} w_j^{(\beta)}u_{i+j-1}^{n+1}\Big] + (1+2\lambda w_1^{(\beta)})u_i^{n+1} = \lambda' g_i^{n+1} + \sum_{j=1}^{n-1}(-b_{j-1} \\ +2b_j - b_{j+1})u_i^{n-j} + (2-b_1)u_i^{n} - b_n u_i^{1} + (2b_n - b_{n-1})u_i^{0} + 2b_n\tau v_2(x_i). \end{aligned} \tag{18} $$

Equations (17) and (18) can be written as

$$ \begin{cases} A_1U^1 = F_1, \\ A_2U^{n+1} = (2-b_1)U^n + \sum_{j=1}^{n-1}(-b_{j-1} + 2b_j - b_{j+1})U^{n-j} - b_nU^1 + (2b_n - b_{n-1})U^0 + F_2, \end{cases} \tag{19} $$

ICCS Camera Ready Version 2021
To cite this paper please use the final published version:
DOI: 10.1007/978-3-030-77961-0_34

where $U^n = \begin{pmatrix} u_1^n \\ u_2^n \\ \vdots \\ u_{N-1}^n \end{pmatrix}$, $F_1 = \lambda^{'} g^1 + 2\tau v_2(x) + 2v_1(x)$, $F_2 = \lambda^{'} g^{n+1} + b_n 2\tau v_2(x)$,

$$A_1 = (a_{i,j}^{(1)})_{(N-1),(N-1)} = \begin{cases} 2(1+\lambda w_1^{(\beta)}), & \text{for } i=j \\ \lambda(w_0^{(\beta)} + w_1^{(\beta)}), & \text{for } j=i-1 \ or \ j=i+1 \\ \lambda w_{i-j+1}^{(\beta)}, & \text{for } j<i-1 \\ \lambda w_{j-i+1}^{(\beta)}, & \text{for } j>i+1, \end{cases}$$

$$A_2 = (a_{i,j}^{(2)})_{(N-1),(N-1)} = \begin{cases} (1+2\lambda w_1^{(\beta)}), & \text{for } i=j \\ \lambda(w_0^{(\beta)} + w_1^{(\beta)}), & \text{for } j=i-1 \ or \ j=i+1 \\ \lambda w_{i-j+1}^{(\beta)}, & \text{for } j<i-1 \\ \lambda w_{j-i+1}^{(\beta)}, & \text{for } j>i+1. \end{cases}$$

4 Stability and convergence

In this section, a theorem [5] is used to proved the stability of implicit finite difference scheme which is discussed in section 3. Furthermore, the convergence of the scheme is also derived. We denote $\|A_1\| = \|A_1\|_\infty = \max_{1\le i\le N-1}\{\sum_{j=1}^{N-1}|a_{i,j}^{(1)}|\}$ and $\|A_2\| = \|A_2\|_\infty = \max_{1\le i\le N-1}\{\sum_{j=1}^{N-1}|a_{i,j}^{(2)}|\}$.

Lemma 2. *$\|A_1^{-1}\| \le 1$ and $\|A_2^{-1}\| \le 1$.*

Proof. We apply the Gerschgorin theorem (see [5] for details)to conclude that every eigenvalue of the matrices A_1 and A_2 have a magnitude strictly larger than 1.
Note that $w_0^{(\beta)} = 1,\ w_1^{(\beta)} = -\beta,\ w_j^{(\beta)} = (-1)^j \frac{\beta(\beta-1)\dots(\beta-j+1)}{j!},\ j = 1,2,3,\dots,$ then for $1<\beta\le 2$ and $j\ge 2$, we have $w_j^{(\beta)} \ge 0$.
We also know that for any $\beta > 0$,

$$(1+z)^\beta = \sum_{m=0}^{\infty} \binom{\beta}{m} z^m, \quad |z|\le 1. \tag{20}$$

Substituting z = -1 in (20) yields $\sum_{j=0}^{\infty} w_j^{(\beta)} = 0$, and then $-w_1^{(\beta)} > \sum_{j=0,j\neq 1}^{i} w_j^{(\beta)}$, i.e. $\sum_{j=0}^{i} w_j^{(\beta)} < 0$ for any $i = 1,2,\cdots m$.

Note that non diagonal elements of matrices A_1 and A_2 are same. According to the Gerschgorin theorem , the eigenvalues of the matrix A_1 lie in the union of $N-1$ circles centered at $a_{i,i}^{(1)}$ with radius $r_i = \sum_{k=1,k\neq i}^{N-1}|a_{i,k}^1|$ and the eigenvalues of the matrix A_2 lie in the union of $N-1$ circles centered at $a_{i,i}^{(2)}$ with the same radius. Using the properties of A_1 and A_2, we have

$$a_{i,i}^{(2)} = 1 + 2\lambda w_1^{(\beta)} = 1 - 2\lambda\beta,\ a_{i,i}^{(1)} = 2 + 2\lambda w_1^{(\beta)} = 2 - 2\lambda\beta,$$

ICCS Camera Ready Version 2021
To cite this paper please use the final published version:
DOI: 10.1007/978-3-030-77961-0_34

where $\lambda = \frac{\tau^{\alpha}\Gamma(3-\alpha)}{h^{\beta}2cos(\beta\pi/2)} < 0$, then we conclude that:

$$\begin{aligned} r_i &= \sum_{k=1,k\neq i}^{N-1} |a_{i,k}^{(1)}| = \sum_{k=1,k\neq i}^{N-1} |a_{i,k}^{(2)}| \\ &\leq 2|\lambda|\beta. \end{aligned}$$

We conclude that eigenvalues of the matrix A_1 and A_2 satisfy $|\rho| \geq 1$ and matrix A_1 and A_2 are diagonally dominant. Then A_1 and A_2 are invertible and eigenvalues are less than or equal to 1 in magnitude. Therefore,

$$\|A_1^{-1}\| \leq \frac{1}{\min\limits_{1\leq i\leq M-1}\{|a_{i,i}^{(1)}| - \sum_{j\neq i,j=1}^{N-1} |a_{i,j}^{(1)}|\}} \leq 1.$$

Similarly, $\|A_2^{-1}\| \leq 1$. □

Now, the stability of implicit finite difference scheme derived in section 3 is proved in the following theorem with help of Lemma 2.

Theorem 3. *The implicit finite difference scheme defined by (19) to the space-time fractional diffusion wave equation (2) with* $1 < \alpha < 2$ *and* $1 < \beta < 2$ *is unconditionally stable.*

Proof. To prove the stability of (19), let $\tilde{u}_i^n$ and u_i^n, $(i = 1, \cdots, N-1;\ n = 1, \cdots, K)$ be the approximate solution of (19). We denote the corresponding error by $\varepsilon_i^n = \tilde{u}_i^n - u_i^n$ and $\varepsilon^n = (\varepsilon_1^n, \cdots, \varepsilon_{N-1}^n)^t$. Then ε^n satisfies if $n = 0$

$$A_1\varepsilon^1 = \varepsilon^0,$$

if $n \geq 0$

$$A_2\varepsilon^{n+1} = \sum_{j=1}^{n-1}(-b_j + 2b_j - b_{j+1})\varepsilon^{n-j} + (2 - b_1)\varepsilon^n + (2b_n - b_{n+1})\varepsilon^0 + b_n\varepsilon^1.$$

Let us prove $\|\varepsilon^n\| \leq C\|\varepsilon^0\|,\ n = 0, 1, 2, \cdots$ by induction, where C is some positive constant. In fact, if $n = 0$

$$\varepsilon^1 = A_1^{-1}\epsilon^0,$$

from that

$$\|\varepsilon^1\| = \|A_1^{-1}\varepsilon^0\| \leq \|A_1^{-1}\|\|\varepsilon^0\|.$$

Since $\|A_1^{-1}\| \leq 1$, from the Lemma (2), we have $\|\varepsilon^1\| \leq \|\varepsilon^0\|$.

Now assume that $\|\varepsilon^s\| \leq C\|\varepsilon^0\|$ for all $s \leq n$, we will prove it is also true for $s = n + 1$. For that we use the properties of the function $f(x) = (x + 1)^{2-\alpha} -$

ICCS Camera Ready Version 2021
To cite this paper please use the final published version:
DOI: 10.1007/978-3-030-77961-0_34

$x^{2-\alpha}(x \geq 0)$, we say that $b_j^{-1} \leq b_n^{-1},\ j = 0, 1, \cdots, n;\ (2b_j - b_{j-1} - b_{j+1}) > 0$ and $(2 - b_1) > 0$, we have

$$\begin{aligned}
\|\varepsilon^{n+1}\| &= \|\sum_{j=1}^{n-1}(-b_j + 2b_j - b_{j+1})A_2^{-1}\varepsilon^{n-j} + (2 - b_1)A_2^{-1}\varepsilon^n + (2b_n - b_{n+1})A_2^{-1}\varepsilon^0 + b_n A_2^{-1}\varepsilon^1\| \\
&\leq \sum_{j=1}^{n-1}(-b_j + 2b_j - b_{j+1})\|A_2^{-1}\varepsilon^{n-j}\| + (2 - b_1)\|A_2^{-1}\varepsilon^n\| + (2b_n - b_{n+1})\|A_2^{-1}\varepsilon^0\| + b_n\|A_2^{-1}\varepsilon^1\| \\
&\leq \sum_{j=1}^{n-1}(-b_j + 2b_j - b_{j+1})\|\varepsilon^{n-j}\| + (2 - b_1)\|\varepsilon^n\| + (2b_n - b_{n+1})\|\varepsilon^0\| + b_n\|\varepsilon^1\| \\
&\leq \Big(\sum_{j=1}^{n-1}(-b_j + 2b_j - b_{j+1}) + (2 - b_1) + (2b_n - b_{n+1}) + b_n\Big)C\|\varepsilon^0\| \\
&\leq C_1\|\varepsilon^0\|.
\end{aligned}$$

Hence the scheme is unconditionally stable. □

Now, for the convergence of implicit finite difference scheme, we proved the following lemma.

Lemma 3. *Suppose that $u(x_i, t_n)$ is the exact solution of (2) at grid point (x_i, t_n), u_i^n is the solution of difference equations (17) and (18), then there exists positive constant M such that*

$$\|e^n\|_\infty \leq b_{n-1}^{-1}M(\tau^3 + \tau^\alpha h), \quad n = 1, 2, \cdots, K,$$

where $\|e^n\|_\infty = \max_{1 \leq i \leq N-1}|e_i^n|$, M is a constant independent of h and τ.

Proof. Define $e_i^n = u(x_i, t_n) - u_i^n,\ i = 1, 2, \cdots, N-1,\ n = 1, 2, \cdots, K$, notice that $e^0 = 0$, we have from (17) and (18) if $n = 0$,

$$\lambda\Big[\sum_{j=0, j\neq 1}^{i+1} w_j^{(\beta)} e_{i-j+1}^1 + \sum_{j=0, j\neq 1}^{N-i+1} w_j^{(\beta)} e_{i+j-1}^1\Big] + 2(1 + \lambda w_1^{(\beta)})e_i^1 = R_i^1,$$

if $n > 0$

$$\begin{aligned}
&\lambda\Big[\sum_{j=0, j\neq 1}^{i+1} w_j^{(\beta)} e_{i-j+1}^{n+1} + \sum_{j=0, j\neq 1}^{N-i+1} w_j^{(\beta)} e_{i+j-1}^{n+1}\Big] + (1 + 2\lambda w_1^{(\beta)})e_i^1 \\
&= \sum_{j=1}^{n-1}(-b_{j-1} + 2b_j + b_{j+1})e_i^{n-j} + (2 - b_1)e_i^n - b_n e_i^1 + R_i^{n+1},
\end{aligned}$$

where $|R_i^{n+1}| \leq M_0(\tau^3 + \tau^\alpha h), i = 1, 2, \cdots, N-1; n = 1, 2, \cdots, K-1$, and M_0 is positive constant independent of τ and h.

ICCS Camera Ready Version 2021
To cite this paper please use the final published version:
DOI: 10.1007/978-3-030-77961-0_34

We use the mathematical induction method to prove the theorem. If $n = 1$, suppose $|e_l^1| = \max\limits_{1\le i\le N-1} |e_i^1|$, and we know that $w_1^{(\beta)} = -\beta$, $\sum_{j=0,j\neq 1}^{N} w_j^{(\beta)} \le \beta$ and $\lambda < 0$, hence

$$\begin{aligned}
|e_l^1| &\le 2(1-\lambda\beta)|e_l^1| + \lambda \sum_{j=0,j\neq 1}^{l+1} w_j^{(\beta)} |e_{l-j+1}^1| + \lambda \sum_{j=0,j\neq 1}^{N-l+1} w_j^{(\beta)} |e_{l+j-1}^1| \\
&\le \big|2(1-\lambda\beta)e_l^1 + \lambda \sum_{j=0,j\neq 1}^{l+1} w_j^{(\beta)} e_{l-j+1}^1 + \lambda \sum_{j=0,j\neq 1}^{N-l+1} w_j^{(\beta)} e_{l+j-1}^1\big| \\
&= |R_l^1| \le C\tau^\alpha(\tau^{3-\alpha}+h) = C\tau^\alpha b_0^{-1}(\tau^{3-\alpha}+h).
\end{aligned}$$

Suppose that if $n \le s$, $\|e^s\|_\infty \le M\tau^\alpha b_{s-1}^{-1}(\tau^{3-\alpha}+h)$ hold, then when $n = s+1$, let $|e_l^{s+1}| = \max\limits_{1\le i\le N-1} |e_i^{s+1}|$. Using the properties of the function $f(x) = (x+1)^{2-\alpha} - x^{2-\alpha}(x \ge 0)$, we say that $b_j^{-1} \le b_n^{-1},\ j = 0, 1, \cdots, n;\ (2b_j - b_{j-1} - b_{j+1}) > 0$ and $(2 - b_1) > 0$, we have

$$\begin{aligned}
|e_l^{s+1}| &\le (1-2\lambda\beta)|e_l^{s+1}| + \lambda \sum_{j=0,j\neq 1}^{l+1} w_j^{(\beta)} |e_{l-j+1}^{s+1}| + \lambda \sum_{j=0,j\neq 1}^{N-l+1} w_j^{(\beta)} |e_{l+j-1}^{s+1}| \\
&\le \big|(1-2\lambda\beta)e_l^{s+1} + \lambda \sum_{j=0,j\neq 1}^{l+1} w_j^{(\beta)} e_{l-j+1}^{s+1} + \lambda \sum_{j=0,j\neq 1}^{N-l+1} w_j^{(\beta)} e_{l+j-1}^{s+1}\big| \\
&\le \sum_{j=1}^{s-1} (2b_j - b_{j-1} - b_{j+1})|e_l^{s-j}| + (2-b_1)|e_l^s| + b_s|e_l^1| + M_0\tau^\alpha(\tau^3+h) \\
&\le M_0 b_s^{-1}\tau^\alpha(\tau^3+h)(b_s + \sum_{j=1}^{s-1}(2b_j - b_{j-1} - b_{j+1}) + (2-b_1) + 1) \\
&\le 3M_0 b_s^{-1}\tau^\alpha(\tau^3+h) = M b_s^{-1}\tau^\alpha(\tau^3+h).
\end{aligned}$$

Thus $\|e^{s+1}\|_\infty \le M b_s^{-1}\tau^\alpha(\tau^{3-\alpha}+h)$. □

Since

$$\lim_{n\to\infty} \frac{b_n^{-1}}{n^\alpha} = \lim_{n\to\infty} \frac{n^{-\alpha}}{(n+1)^{2-\alpha} - n^{2-\alpha}} = \frac{1}{2-\alpha}.$$

Hence, there exists constant $C > 0$, such that

$$\|e^n\|_\infty \le n^\alpha C(\tau^3 + \tau^\alpha h) = (n\tau)^\alpha C(\tau^{3-\alpha}+h), \quad n = 1, 2, \cdots, K.$$

When $n\tau \le T$, we get the following theorem:

Theorem 4. *Suppose that $u(x_i, t_n)$ is the exact solution of (2) at grid point (x_i, t_n), u_i^n is the solution of difference equations (17) and (18), then there exists positive constant C, such that*

$$|u(x_i, t_n) - u_i^n| \le C(\tau^{3-\alpha}+h),\ i = 1, 2, \cdots, N-1;\ n = 1, 2, \cdots, K.$$

ICCS Camera Ready Version 2021
To cite this paper please use the final published version:
DOI: 10.1007/978-3-030-77961-0_34

5 Numerical Results

In this section, we present the numerical results of space-time fractional wave equation to illustrate the presented scheme that is employed in our study. We take

$$u(x,t) = (t^{2+\alpha} + t + 2)x^4(1-x)^4,$$

is exact solution of (2) which implies that $v_1(x) = 2x^4(1-x)^4,\ v_2(x) = x^4(1-x)^4,\ g(x,t) = B + C + D$, where,

$$B = x^4(1-x)^4\left[\frac{\Gamma(3+\alpha)}{\Gamma(3)}t^2\right],$$

$$C = \frac{(t^{2+\alpha}+t+2)}{2cos(\beta\pi/2)}\left[\frac{\Gamma(9)}{\Gamma(9-\beta)}x^{8-\beta} - \frac{4\Gamma(8)}{\Gamma(8-\beta)}x^{7-\beta} + \frac{6\Gamma(7)}{\Gamma(7-\beta)}x^{6-\beta} - \frac{4\Gamma(6)}{\Gamma(6-\beta)}x^{5-\beta} + \frac{\Gamma(5)}{\Gamma(5-\beta)}x^{4-\beta}\right],$$

$$D = \frac{(t^{2+\alpha}+t+2)}{2cos(\beta\pi/2)}\left[\frac{\Gamma(9)}{\Gamma(9-\beta)}(1-x)^{8-\beta} - \frac{4\Gamma(8)}{\Gamma(8-\beta)}(1-x)^{7-\beta} + \frac{6\Gamma(7)}{\Gamma(7-\beta)}(1-x)^{6-\beta} - \frac{4\Gamma(6)}{\Gamma(6-\beta)}(1-x)^{5-\beta} + \frac{\Gamma(5)}{\Gamma(5-\beta)}(1-x)^{4-\beta}\right].$$

We measure the absolute maximum error between exact solution and the finite difference approximation U^K. The numerical results by using implicit finite difference scheme with $\alpha = 1.5$, $\tau = 0.001$ and $t_K = T = 1$ are presented in Table 1.

Table 1: The absolute maximum error and convergence rates of the implicit finite difference approximation (19) with $\alpha = 1.5$, $t_K = 1$ and $\tau = 0.001$

1/h	$\beta = 1.2$	Rate	$\beta = 1.5$	Rate	$\beta = 1.8$	Rate
8	0.005900	-	0.001500	-	0.000352	-
16	0.003800	0.6347	0.000947	0.6634	0.000073	2.2772
32	0.002200	0.7885	0.000529	0.8412	0.000042	0.8101
64	0.001200	0.8745	0.000278	0.9286	0.000029	0.5137
128	0.000617	0.9597	0.000141	0.9761	0.000016	0.8786
256	0.000314	0.9745	0.000070	1.0103	0.000007	1.1098
512	0.000157	0.9940	0.000034	1.0503	0.000003	1.2224

We have proved in Section 5 that proposed implicit finite difference scheme is first order accurate in spatial variable. In order to show the convergence rate

ICCS Camera Ready Version 2021
To cite this paper please use the final published version:
DOI: 10.1007/978-3-030-77961-0_34

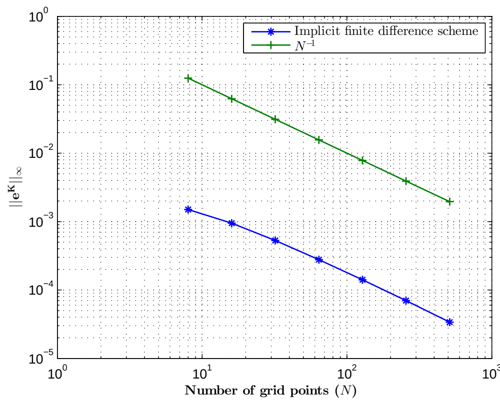

Fig. 1: Absolute maximum error between the numerical solution and exact solution for $\alpha = 1.5,\ \beta = 1.5,\ \text{and}\ T = 1$

numerically, we take $\tau = 0.001$. In Figure 1, absolute maximum error between the numerical solution and exact solution for $\alpha = 1.5,\ \beta = 1.5,\ \text{and}\ T = 1$ is plotted with respect to the number of grid points. It can be observed from Figure 1 that the proposed implicit finite difference scheme exhibits approximately first order convergence rate.

6 Conclusion

In this paper, we provide the implicit finite difference scheme for solving the space-time fractional wave equation as described in (2). The unconditional stability, the rate of convergence and the error estimate of the implicit finite difference scheme are discussed and proved rigorously. The performance of the new scheme is investigated through a numerical example with a known exact solution. From the obtained numerical results in Table 1, we conclude that the numerical solution is in excellent agreement with the exact solution when our scheme is employed to the space-time fractional wave equation as described in (2).

References

1. Chen, A., Li, C.: Numerical solution of fractional diffusion-wave equation. Numerical Functional Analysis and Optimization **37**(1), 19–39 (2016)
2. Fino, A.Z., Ibrahim, H.: Analytical solution for a generalized space-time fractional telegraph equation. Mathematical Methods in the Applied Sciences **36**(14), 1813–1824 (2013)
3. Hilfer, R.: Fractional calculus and regular variation in thermodynamics. World Scientific (2000)

ICCS Camera Ready Version 2021
To cite this paper please use the final published version:
DOI: 10.1007/978-3-030-77961-0_34

4. Iomin, A., Dorfman, S., Dorfman, L.: On tumor development: fractional transport approach. arXiv preprint q-bio/0406001 (2004)
5. Isaacson, E., Keller, H.B.: Analysis of Numerical Methods. John wiley and sons., New York (1966)
6. Khader, M.M., Adel, M.H.: Numerical solutions of fractional wave equations using an efficient class of fdm based on the hermite formula. Advances in Difference Equations **2016**(1), 34 (2016)
7. Kilbas, A., Srivastava, H., Trujillo, J.: Theory and applications of fractional differential equations. Elsevier Science Limited (2006)
8. Li, C., Zeng, F.: Numerical methods for fractional calculus. Chapman and Hall/CRC (2015)
9. Luchko, Y., Gorenflo, R.: An operational method for solving fractional differential equations with the caputo derivatives. Acta Math. Vietnam **24**(2), 207–233 (1999)
10. Mainardi, F., Paradisi, P.: A model of diffusive waves in viscoelasticity based on fractional calculus. Decision and Control, Proceedings of the 36th IEEE Conference **5**, 4961–4966 (1997)
11. Meerschaert, M.M., Scalas, E.: Coupled continuous time random walks in finance. Physica A: Statistical Mechanics and its Applications **370**(1), 114–118 (2006)
12. Meerschaert, M.M., Zhang, Y., Baeumer, B.: Particle tracking for fractional diffusion with two time scales. Computers and mathematics with applications **59**(3), 1078–1086 (2010)
13. Podlubny, I.: Fractional differential equations, vol. 198
14. Raberto, M., Scalas, E., Mainardi, F.: Waiting-times and returns in high-frequency financial data: an empirical study. Physica A: Statistical Mechanics and its Applications **314**(1-4), 749–755 (2002)
15. Singh, A.K., Mehra, M.: Uncertainty quantification in fractional stochastic integro-differential equations using legendre wavelet collocation method. In: Krzhizhanovskaya V. et al. (eds) Computational Science – ICCS 2020. Lecture Notes in Computer Science. vol. 12138, pp. 58–71. Springer (2020)
16. Singh, A.K., Mehra, M.: Wavelet collocation method based on legendre polynomials and its application in solving the stochastic fractional integro-differential equations. Journal of Computational Science p. https://doi.org/10.1016/j.jocs.2021.101342 (2021)
17. Sun, Z.z., Wu, X.: A fully discrete difference scheme for a diffusion-wave system. Applied Numerical Mathematics **56**(2), 193–209 (2006)
18. Sweilam, N.H., Khader, M.M., Nagy, A.: Numerical solution of two-sided space-fractional wave equation using finite difference method. Journal of Computational and Applied Mathematics **235**(8), 2832–2841 (2011)
19. Zhang, Y.n., Sun, Z.z., Zhao, X.: Compact alternating direction implicit scheme for the two-dimensional fractional diffusion-wave equation. SIAM Journal on Numerical Analysis **50**(3), 1535–1555 (2012)

ICCS Camera Ready Version 2021
To cite this paper please use the final published version:
DOI: 10.1007/978-3-030-77961-0_34