Design of Short Codes for Quantum Channels with Asymmetric Pauli Errors*

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Abstract. One of the main problems in quantum information systems is the presence of errors due to noise. Many quantum error correcting codes have been designed to deal with generic errors. In this paper we construct new stabilizer codes able to correct a given number e_{g} of generic Pauli $\boldsymbol{X}, \boldsymbol{Y}$ and $\boldsymbol{Z}$ errors, plus a number e_{Z} of Pauli errors of a specified type (e.g., $\boldsymbol{Z}$ errors). These codes can be of interest when the quantum channel is asymmetric, i.e., when some types of error occur more frequently than others. For example, we design a $[[9, 1]]$ quantum error correcting code able to correct up to one generic qubit error plus one $\boldsymbol{Z}$ error in arbitrary positions. According to a generalized version of the quantum Hamming bound, it is the shortest code with this error correction capability.

1 Introduction

The possibility to exploit the unique features of quantum mechanics is paving the way to new approaches for acquiring, processing and transmitting information, with applications in quantum communications, computing, cryptography, and sensing [1–10]. In this regard, one of the main problem is the noise due to unwanted interaction of the quantum information with the environment. Error correction techniques are therefore essential for quantum computation, quantum memories and quantum communication systems [11–14]. Compared to the classical case, quantum error correction is made more difficult by the laws of quantum mechanics which imply that qubits cannot be copied or measured without perturbing state superposition [15]. Moreover, there is continuum of errors that could occur on a qubit. However, it has been shown that in order to correct an arbitrary qubit error it suffices to consider error correction on the discrete set of Pauli operators, i.e., the bit flip $\boldsymbol{X}$, phase flip $\boldsymbol{Z}$, and combined bit-phase flip $\boldsymbol{Y}$ [11, 16–18]. Hence, we can consider in general a channel introducing qubit errors $\boldsymbol{X}$, $\boldsymbol{Y}$, and $\boldsymbol{Z}$ with probabilities p_{x}, p_{y}, and p_{z}, respectively, and leaving the qubit intact with probability $1 - \rho$, where $\rho = p_{\mathrm{x}} + p_{\mathrm{y}} + p_{\mathrm{z}}$. A special case of this model is the so-called depolarizing channel for which $p_{\mathrm{x}} = p_{\mathrm{y}} = p_{\mathrm{z}} = \rho/3$. Quantum error correcting codes for this channel are naturally designed to protect against equiprobable Pauli errors [19–21].

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However, not all channels exhibit this symmetric behaviour of Pauli errors as, in some situations, some types of error are more likely than others [22]. In fact, depending on the technology adopted for the system implementation, the different types of Pauli error can have quite different probabilities of occurrence, leading to asymmetric quantum channels [23, 24]. An example is the Pauli-twirled channel associated to the combination of amplitude damping and dephasing channels [23]. This model has $p_{\mathrm{x}} = p_{\mathrm{y}}$ and $p_{\mathrm{z}} = A\rho/(A+2)$, where ρ is the error probability, and the asymmetry is accounted for by the parameter $A = p_{\mathrm{z}}/p_{\mathrm{x}}$. This parameter is a function of the relaxation time, T_1, and the dephasing time, T_2, which are in general different, leading to $A > 1$ [24,25].

Owing to this considerations, it can be useful to investigate the design of quantum codes with error correction capabilities tailored to specific channel models. For example, codes for the amplitude damping channel have been proposed in [26–31], while quantum error correcting codes for more general asymmetric channels are investigated in [22–25]. In particular, asymmetric Calderbank Shor Steane (CSS) codes, where the two classical parity check matrices are chosen with different error correction capability (e.g., Bose Chaudhuri Hocquenghem (BCH) codes for $\boldsymbol{X}$ errors and low density parity check (LDPC) codes for $\boldsymbol{Z}$ errors), are investigated in [22, 23]. Inherent to the CSS construction there are two distinct error correction capabilities for the $\boldsymbol{X}$ and the $\boldsymbol{Z}$ errors; the resulting asymmetric codes, denoted as $[[n, k, d_{\mathrm{x}}/d_{\mathrm{z}}]]$, can correct up to $t_{\mathrm{x}} = \lfloor (d_{\mathrm{x}} - 1)/2 \rfloor$ Pauli $\boldsymbol{X}$ errors and $t_{\mathrm{z}} = \lfloor (d_{\mathrm{z}} - 1)/2 \rfloor$ Pauli $\boldsymbol{Z}$ errors per codeword. In fact, due to the possibility of employing tools from classical error correction, many works have been focused on asymmetric codes based on the CSS construction, which, however, may not lead to the shortest codes (e.g., for the symmetric channel compare the $[[7, 1]]$ CSS Steane code with the shortest $[[5, 1]]$ code [20, 21]).

In this paper, we relax the CSS constraint in order to obtain the shortest asymmetric stabilizer codes able to correct a given number e_{g} of generic Pauli errors, plus a number e_{Z} of Pauli errors of a specified type (e.g., $\boldsymbol{Z}$ errors). We denote these as the asymmetric $[[n, k]]$ codes with $(e_{\mathrm{g}}, e_{\mathrm{Z}})$. To this aim we first derive a generalized version of the quantum Hamming bound, which was developed to correct generic errors, for an asymmetric error correction capability $(e_{\mathrm{g}}, e_{\mathrm{Z}})$. Then, we construct a $[[9, 1]]$ code with $(e_{\mathrm{g}} = 1, e_{\mathrm{Z}} = 1)$ which, according to the new quantum Hamming bound, is the shortest possible code. Finally, we extend the construction method to the class of $[[n, 1]]$ codes with $e_{\mathrm{g}} = 1$ and arbitrary e_{Z}.

2 Notation

A qubit is an element of the two-dimensional Hilbert space $\mathcal{H}^2$, with basis $|0\rangle$ and $|1\rangle$ [32]. An n-tuple of qubits (n qubits) is an element of the 2^n-dimensional Hilbert space, $\mathcal{H}^{2^n}$, with basis composed by all possible tensor products $|i_1\rangle\,|i_2\rangle \cdots |i_n\rangle$, with $i_j \in \{0, 1\}, 1 \le j \le n$. The Pauli operators, denoted as $\boldsymbol{I}, \boldsymbol{X}, \boldsymbol{Z}$, and $\boldsymbol{Y}$, are defined by $\boldsymbol{I}\,|a\rangle = |a\rangle$, $\boldsymbol{X}\,|a\rangle = |a \oplus 1\rangle$, $\boldsymbol{Z}\,|a\rangle = (-1)^a\,|a\rangle$, and $\boldsymbol{Y}\,|a\rangle = i(-1)^a\,|a \oplus 1\rangle$ for $a \in \{0, 1\}$. These operators either commute or anticommute.

With $[[n, k]]$ we indicate a quantum error correcting code (QECC) that encodes k data qubits $|\varphi\rangle$ into a codeword of n qubits $|\psi\rangle$. We use the stabilizer formalism, where a

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stabilizer code $\mathcal{C}$ is generated by $n-k$ independent and commuting operators $\boldsymbol{G}_i \in \mathcal{G}_n$, called generators [32–34]. The code $\mathcal{C}$ is the set of quantum states $|\psi\rangle$ satisfying

$$\boldsymbol{G}_i |\psi\rangle = |\psi\rangle \ , \quad i = 1, 2, \ldots, n-k \, . \tag{1}$$

Assume a codeword $|\psi\rangle \in \mathcal{C}$ affected by a channel error described by the operator $\boldsymbol{E} \in \mathcal{G}_n$. For error correction, the received state $\boldsymbol{E}|\psi\rangle$ is measured according to the generators $\boldsymbol{G}_1, \boldsymbol{G}_2, \ldots, \boldsymbol{G}_{n-k}$, resulting in a quantum error syndrome $\boldsymbol{s}(\boldsymbol{E}) = (s_1, s_2, \ldots, s_{n-k})$, with each $s_i = 0$ or 1 depending on the fact that $\boldsymbol{E}$ commutes or anticommutes with $\boldsymbol{G}_i$, respectively. Note that, due to (1), the syndrome depends on $\boldsymbol{E}$ and not on the particular q-codeword $|\psi\rangle$. Moreover, measuring the syndrome does not change the quantum state, which remains $\boldsymbol{E}|\psi\rangle$. Let $\mathcal{S} = \{\boldsymbol{s}^{(1)}, \boldsymbol{s}^{(2)}, \ldots, \boldsymbol{s}^{(m)}\}$ be the set of $m = 2^{n-k}$ possible syndromes, with $\boldsymbol{s}^{(1)} = (0, 0, \ldots, 0)$ denoting the syndrome of the operators $\boldsymbol{E}$ (including the identity $\boldsymbol{I}$, i.e., the no-errors operator) such that $\boldsymbol{E}|\psi\rangle$ is still a valid q-codeword. A generic Pauli error $\boldsymbol{E} \in \mathcal{G}_n$ can be described by specifying the single Pauli errors on each qubit. We use $\boldsymbol{X}_i, \boldsymbol{Y}_i, \boldsymbol{Z}_i$ to denote the Pauli $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}$ error on the i-th qubit.

3 Hamming Bounds for Quantum Asymmetric Codes

The standard quantum Hamming bound (QHB) gives a necessary condition for the existence of non-degenerate error correcting codes: a quantum code which encodes k qubits in n qubits can correct up to t generic errors per codeword only if [17, 35]

$$2^{n-k} \geq \sum_{j=0}^{t} \binom{n}{j} 3^j \, . \tag{2}$$

The bound is easily proved by noticing that the number of syndromes, 2^{n-k}, must be at least equal to that of the distinct errors we want to correct. Since for each position there could be three Pauli errors ($\boldsymbol{X}$, $\boldsymbol{Y}$ or $\boldsymbol{Z}$), the number of distinct patterns having j qubits in error is $\binom{n}{j}3^j$, and this gives the bound (2).

In this paper we investigate non-degenerate QECCs which can correct some generic errors ($\boldsymbol{X}$, $\boldsymbol{Y}$ or $\boldsymbol{Z}$), plus some fixed errors (e.g., $\boldsymbol{Z}$ errors). We derive therefore the following generalized quantum Hamming bound (GQHB).

Theorem 1 (Generalized Quantum Hamming Bound). *A quantum code which encodes k qubits in n qubits can correct up to e_{g} generic errors plus up to e_{Z} fixed errors per codeword only if*

$$2^{n-k} \geq \sum_{j=0}^{e_{\mathrm{g}}+e_{\mathrm{Z}}} \binom{n}{j} \sum_{i=0}^{e_{\mathrm{g}}} \binom{j}{i} 2^i \, . \tag{3}$$

Proof. For the proof we need to enumerate the different patterns of error. The number of patterns of up to e_{g} generic errors is given by (2) with $t = e_{\mathrm{g}}$. Then, we have to add the number of configurations with $e_{\mathrm{g}} < j \leq e_{\mathrm{g}} + e_{\mathrm{Z}}$ errors, composed by e_{g} generic

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errors and the remaining $j - e_\mathrm{g}$ Pauli $\boldsymbol{Z}$ errors. We can write

$$2^{n-k} \geq \sum_{j=0}^{e_\mathrm{g}} \binom{n}{j} 3^j + \sum_{j=e_\mathrm{g}+1}^{e_\mathrm{g}+e_\mathrm{Z}} \binom{n}{j} \left[3^j - f(j; e_\mathrm{g})\right] \tag{4}$$

where $f(j; e_\mathrm{g})$ is a function that returns the number of non-correctable patterns of j errors. This is the solution of the following combinatorial problem: given j positions of the errors, count the number of all combinations with more than e_g symbols from the set $\mathcal{P}_\mathrm{XY} = \{\boldsymbol{X}, \boldsymbol{Y}\}$ and the remaining from the set $\mathcal{P}_\mathrm{Z} = \{\boldsymbol{Z}\}$. We have therefore

$$f(j; e_\mathrm{g}) = \sum_{i=0}^{j-e_\mathrm{g}-1} \binom{j}{i} 2^{j-i} \tag{5}$$

which allows to write

$$\begin{aligned} g(j; e_\mathrm{g}) &= 3^j - f(j; e_\mathrm{g}) \\ &= \sum_{i=0}^{j} \binom{j}{i} 2^{j-i} - \sum_{i=0}^{j-e_\mathrm{g}-1} \binom{j}{i} 2^{j-i} \\ &= \sum_{i=j-e_\mathrm{g}}^{j} \binom{j}{i} 2^{j-i} = \sum_{i=0}^{e_\mathrm{g}} \binom{j}{i} 2^{i}\,. \end{aligned} \tag{6}$$

It is easy to see that $g(j; e_\mathrm{g})$ is equal to 3^j if $j \leq e_\mathrm{g}$, so substituting and incorporating the summation in (4) we finally obtain

$$2^{n-k} \geq \sum_{j=0}^{e_\mathrm{g}+e_\mathrm{Z}} \binom{n}{j} g(j; e_\mathrm{g}) = \sum_{j=0}^{e_\mathrm{g}+e_\mathrm{Z}} \binom{n}{j} \sum_{i=0}^{e_\mathrm{g}} \binom{j}{i} 2^{i}\,. \tag{7}$$

The GQHB can be used to compare codes which can correct t generic errors with codes correcting a total of t errors with e_g of them generic and the others fixed. In Table 1 we report the minimum code lengths n_min resulting from the Hamming bounds, for different values of the total number of errors t, and assuming $e_\mathrm{g} = 1$ for the GQHB. From the table we can observe the possible gain in qubits for the asymmetric case.

	$t = 1$	$t = 2$	$t = 3$	$t = 4$
$k = 1$	5,5	10,9	15,12	20,15
$k = 2$	7,7	12,10	16,14	21,17
$k = 3$	8,8	13,12	18,15	23,19

Table 1: Comparison between the minimum code lengths $n_\mathrm{min}^\mathrm{QHB}, n_\mathrm{min}^\mathrm{GQHB}$ according to the Hamming bounds (2), (3). For the GQHB $t = e_\mathrm{g} + e_\mathrm{Z}$ with $e_\mathrm{g} = 1$.

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4 Construction of short asymmetric codes by syndrome assignment

In this section we present a construction of short stabilizer asymmetric codes with $k = 1$ and $e_{\mathrm{g}} = 1$, i.e., for $[[n, 1]]$ QECCs with error correction capability $(1, e_{\mathrm{Z}})$. The design is based on the error syndromes: specifically, we proceed by assigning different syndromes to the different correctable error patters.

Let us first observe that the vector syndrome of a composed error $\boldsymbol{E} = \boldsymbol{E}_1\boldsymbol{E}_2$, with $\boldsymbol{E}_1, \boldsymbol{E}_2 \in \mathcal{G}_n$, can be expressed as $\boldsymbol{s}(\boldsymbol{E}) = \boldsymbol{s}(\boldsymbol{E}_1\boldsymbol{E}_2) = \boldsymbol{s}(\boldsymbol{E}_2\boldsymbol{E}_1) = \boldsymbol{s}(\boldsymbol{E}_1) \oplus \boldsymbol{s}(\boldsymbol{E}_2)$ where $\oplus$ is the elementwise modulo 2 addition. Moreover, $\boldsymbol{X}\boldsymbol{Z} = i\boldsymbol{Y}$, and for the syndromes we have $\boldsymbol{s}(\boldsymbol{X}_i\boldsymbol{Z}_i) = \boldsymbol{s}(\boldsymbol{Y}_i)$, $\boldsymbol{s}(\boldsymbol{X}_i\boldsymbol{Y}_i) = \boldsymbol{s}(\boldsymbol{Z}_i)$, and $\boldsymbol{s}(\boldsymbol{Y}_i\boldsymbol{Z}_i) = \boldsymbol{s}(\boldsymbol{X}_i)$. Hence, once we have assigned the syndromes for the single error patterns $\boldsymbol{X}_i$ and $\boldsymbol{Z}_i$, with $i = 1, \ldots, n$, the syndromes for all possible errors are automatically determined.

The key point in the design is therefore to find an assignment giving distinct syndromes for all correctable error patterns.

In the following, if not specified otherwise, the indexes i, j will run from 1 to n, and the index ℓ will run from 1 to $n - 1$. Also, the weight of a syndrome is the number of non-zero elements in the associated vector.

4.1 Construction of [[n, 1]] QECCs with $e_{\mathrm{g}} = 1, e_{\mathrm{Z}} = 1$

For this case we need to solve the following problem: assign $2n$ syndromes $\boldsymbol{s}(\boldsymbol{X}_i)$ and $\boldsymbol{s}(\boldsymbol{Z}_i)$ such that the syndromes of the errors $\boldsymbol{I}$, $\boldsymbol{X}_i$, $\boldsymbol{Y}_i$, $\boldsymbol{Z}_i$, $\boldsymbol{X}_i\boldsymbol{Z}_j$, $\boldsymbol{Y}_i\boldsymbol{Z}_j$, $\boldsymbol{Z}_i\boldsymbol{Z}_j$, $\forall i \neq j$, are all different.
Now, we aim to construct the shortest possible code according to the GQHB, i.e., a code with $n = 9$ (see Table 1). We start by assigning the syndromes of $\boldsymbol{Z}_i$ as reported in the following table.

	s_8	s_7	s_6	s_5	s_4	s_3	s_2	s_1
$\boldsymbol{Z}_1$	0	0	0	0	0	0	0	1
$\boldsymbol{Z}_2$	0	0	0	0	0	0	1	0
$\boldsymbol{Z}_3$	0	0	0	0	0	1	0	0
$\boldsymbol{Z}_4$	0	0	0	0	1	0	0	0
$\boldsymbol{Z}_5$	0	0	0	1	0	0	0	0
$\boldsymbol{Z}_6$	0	0	1	0	0	0	0	0
$\boldsymbol{Z}_7$	0	1	0	0	0	0	0	0
$\boldsymbol{Z}_8$	1	0	0	0	0	0	0	0
$\boldsymbol{Z}_9$	1	1	1	1	1	1	1	1

Table 2: Assigned syndromes for single Pauli $\boldsymbol{Z}$ errors.

With this choice we have assigned all possible syndromes of weight 1 and 8. Also, the combinations of $\boldsymbol{Z}_i\boldsymbol{Z}_j$ with $i \neq j$, cover all possible syndromes of weight 2 and 7.

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To assign the syndromes of $\boldsymbol{X}_i$ we then use a Monte Carlo approach. To reduce the search space, i.e., the set of possible syndromes, we observe the following:

- The weight of $\boldsymbol{s}(\boldsymbol{X}_i)$ cannot be 3 or 6. This is because otherwise $\boldsymbol{s}(\boldsymbol{Z}_j\boldsymbol{X}_i)$ would have weight 2 or 7 for some i and j, which are already assigned for errors of the type $\boldsymbol{Z}_i\boldsymbol{Z}_j$. Therefore the possible weights for $\boldsymbol{s}(\boldsymbol{X}_i)$ are only 4 and 5. The same observation applies to $\boldsymbol{s}(\boldsymbol{Y}_i)$. We then fix the weight for $\boldsymbol{s}(\boldsymbol{X}_i)$ equal to 4.
- We can obtain $\boldsymbol{s}(\boldsymbol{Y}_\ell)$ with weight 5 for $\ell = 1, \dots, 8$, by imposing to "0" the ℓ-th element of the syndrome of $\boldsymbol{X}_\ell$. Note that $\boldsymbol{Y}_9$ has weight 4 since $\boldsymbol{X}_9$ has weight 4.

By following the previous rules, a possible assignment obtained by Monte Carlo is reported in Table 3.

	s_8	s_7	s_6	s_5	s_4	s_3	s_2	s_1
$\boldsymbol{X}_1$	1	0	1	1	1	0	0	0
$\boldsymbol{X}_2$	1	0	0	1	0	1	0	1
$\boldsymbol{X}_3$	0	0	1	0	1	0	1	1
$\boldsymbol{X}_4$	1	1	1	0	0	1	0	0
$\boldsymbol{X}_5$	0	1	0	0	1	1	0	1
$\boldsymbol{X}_6$	1	1	0	0	0	0	1	1
$\boldsymbol{X}_7$	0	0	1	1	0	1	1	0
$\boldsymbol{X}_8$	0	1	0	1	1	0	1	0
$\boldsymbol{X}_9$	1	0	0	0	1	1	1	0

Table 3: Possible syndromes for single Pauli $\boldsymbol{X}$ errors.

From Table 2 and Table 3 we can then build the stabilizer matrix with the following procedure, where $s_j(\boldsymbol{X}_i)$ indicates the j-th elements of the $\boldsymbol{X}_i$'s syndrome:

- if $s_j(\boldsymbol{X}_i) = 0$ and $s_j(\boldsymbol{Z}_i) = 0$ put the element $\boldsymbol{I}$ in position (j, i) of the stabilizer matrix because it is the only Pauli operator which commutes with both.
- if $s_j(\boldsymbol{X}_i) = 1$ and $s_j(\boldsymbol{Z}_i) = 0$ put the element $\boldsymbol{Z}$ in position (j, i) of the stabilizer matrix because it is the only Pauli operator which commutes with $\boldsymbol{Z}$ and anti-commute with $\boldsymbol{X}$.
- if $s_j(\boldsymbol{X}_i) = 0$ and $s_j(\boldsymbol{Z}_i) = 1$ put the element $\boldsymbol{X}$ in position (j, i) of the stabilizer matrix because it is the only Pauli operator which commutes with $\boldsymbol{X}$ and anti-commute with $\boldsymbol{Z}$.
- if $s_j(\boldsymbol{X}_i) = 1$ and $s_j(\boldsymbol{Z}_i) = 1$ put the element $\boldsymbol{Y}$ in position (j, i) of the stabilizer matrix because it is the only Pauli operator which anti-commutes with both.

The resulting stabilizer matrix, after checking the commutation conditions, is represented in Table 4.

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	1	2	3	4	5	6	7	8	9
$\boldsymbol{G}_1$	$\boldsymbol{X}$	$\boldsymbol{Z}$	$\boldsymbol{Z}$	$\boldsymbol{I}$	$\boldsymbol{Z}$	$\boldsymbol{Z}$	$\boldsymbol{I}$	$\boldsymbol{I}$	$\boldsymbol{X}$
$\boldsymbol{G}_2$	$\boldsymbol{I}$	$\boldsymbol{X}$	$\boldsymbol{Z}$	$\boldsymbol{I}$	$\boldsymbol{I}$	$\boldsymbol{Z}$	$\boldsymbol{Z}$	$\boldsymbol{Z}$	$\boldsymbol{Y}$
$\boldsymbol{G}_3$	$\boldsymbol{I}$	$\boldsymbol{Z}$	$\boldsymbol{X}$	$\boldsymbol{Z}$	$\boldsymbol{Z}$	$\boldsymbol{I}$	$\boldsymbol{Z}$	$\boldsymbol{I}$	$\boldsymbol{Y}$
$\boldsymbol{G}_4$	$\boldsymbol{Z}$	$\boldsymbol{I}$	$\boldsymbol{Z}$	$\boldsymbol{X}$	$\boldsymbol{Z}$	$\boldsymbol{I}$	$\boldsymbol{I}$	$\boldsymbol{Z}$	$\boldsymbol{Y}$
$\boldsymbol{G}_5$	$\boldsymbol{Z}$	$\boldsymbol{Z}$	$\boldsymbol{I}$	$\boldsymbol{I}$	$\boldsymbol{X}$	$\boldsymbol{I}$	$\boldsymbol{Z}$	$\boldsymbol{Z}$	$\boldsymbol{X}$
$\boldsymbol{G}_6$	$\boldsymbol{Z}$	$\boldsymbol{I}$	$\boldsymbol{Z}$	$\boldsymbol{Z}$	$\boldsymbol{I}$	$\boldsymbol{X}$	$\boldsymbol{Z}$	$\boldsymbol{I}$	$\boldsymbol{X}$
$\boldsymbol{G}_7$	$\boldsymbol{I}$	$\boldsymbol{I}$	$\boldsymbol{I}$	$\boldsymbol{Z}$	$\boldsymbol{Z}$	$\boldsymbol{Z}$	$\boldsymbol{X}$	$\boldsymbol{Z}$	$\boldsymbol{X}$
$\boldsymbol{G}_8$	$\boldsymbol{Z}$	$\boldsymbol{Z}$	$\boldsymbol{I}$	$\boldsymbol{Z}$	$\boldsymbol{I}$	$\boldsymbol{Z}$	$\boldsymbol{I}$	$\boldsymbol{X}$	$\boldsymbol{Y}$

Table 4: Stabilizer for a $[[9,1]]$ QECC with $e_{\mathrm{g}} = 1$ and $e_{\mathrm{Z}} = 1$.

4.2 Construction of $[[n, 1]]$ QECCs with $e_{\mathrm{g}} = 1$ and $e_{\mathrm{Z}} \geqslant 1$

The construction presented in the previous section can be generalized to the case of more fixed errors, $e_{\mathrm{Z}} \geqslant 1$. In this section we indicate $\tilde{t} = e_{\mathrm{g}} + e_{\mathrm{Z}}$. By adopting the same assignment proposed in Table 2, it is easy to see that we use all possible syndromes with weight in the range $\left[0, \tilde{t}\right]$ and $\left[n-\tilde{t}, n-1\right]$, covering all possible error operators with up to $\tilde{t}$ errors of type $\boldsymbol{Z}$. For the assignment of the syndromes $\boldsymbol{s}(\boldsymbol{X}_i)$ we can generalize the previously exposed arguments, as follows:

- The weight of $\boldsymbol{s}(\boldsymbol{X}_i)$ cannot be less than $2\tilde{t}$ or greater than $n - 2\tilde{t}$. This is because otherwise $\boldsymbol{s}(\boldsymbol{Z}_{j_1} \ldots \boldsymbol{Z}_{j_L} \boldsymbol{X}_i)$ would have weight in the range $\left[0, \tilde{t}\right]$ or $\left[n-\tilde{t}, n-1\right]$ for some $L \leq e_{\mathrm{Z}}$ and some choices of $j_1, \ldots, j_{\mathrm{L}}$. These syndromes are already assigned for errors of the type $\boldsymbol{Z}_{j_1} \ldots \boldsymbol{Z}_{j_M}$ for some $M \leq e_{\mathrm{Z}}$ and some choices of $j_1, \ldots, j_{\mathrm{M}}$. Therefore the possible weights for $\boldsymbol{s}(\boldsymbol{X}_i)$ are in the range $\left[2\tilde{t}, n - 2\tilde{t}\right]$. The same observation applies to $\boldsymbol{s}(\boldsymbol{Y}_i)$.
- Setting the ℓ-th element of the syndrome of $\boldsymbol{X}_\ell$ to "0" we obtain that $\boldsymbol{s}(\boldsymbol{Y}_\ell)$ has the weight of $\boldsymbol{s}(\boldsymbol{X}_\ell)$ increased by 1, with $\ell = 1, \ldots, n-1$. Hence, in order to have both $\boldsymbol{s}(\boldsymbol{X}_\ell)$ and $\boldsymbol{s}(\boldsymbol{Y}_\ell)$ in the permitted range, we must have $n - 4\tilde{t} \geq 1$. Note that this constraint can be stricter than the GQHB. For example, we cannot construct the $[[12,1]]$ code with $e_{\mathrm{g}} = 1, e_{\mathrm{Z}} = 2$.
- With the previous choice, the sum of the weights of the syndromes $\boldsymbol{s}(\boldsymbol{Y}_n)$ and $\boldsymbol{s}(\boldsymbol{X}_n)$ is $n-1$. Then, a good choice is to assign to $\boldsymbol{s}(\boldsymbol{X}_n)$ a weight $\lceil (n-1)/2 \rceil$ or $\lfloor (n-1)/2 \rfloor$. In this case, if n is odd $\boldsymbol{s}(\boldsymbol{Y}_n)$ would have the same weight, which is in the correct range because $n - 4\tilde{t} \geq 0$ is guaranteed by the previous point; if n is even the weights are still in the correct range because $n - 4\tilde{t} \geq 1$.

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The resulting algorithm is reported below.

Result: Stabilizer matrix from Assignment Construction
Choose n and $\tilde{t}$ to satisfy the constraint $n - 4\tilde{t} \geq 1$;
Assign $\boldsymbol{s}(\boldsymbol{Z}_i)$ as in Table 2;
Pick a random syndrome for $\boldsymbol{s}(\boldsymbol{X}_n)$ with weight $\lfloor (n-1)/2 \rfloor$;
Assign $\boldsymbol{s}(\boldsymbol{Y}_n)$, $\boldsymbol{s}(\boldsymbol{X}_n \boldsymbol{Z}_{j_1} \ldots \boldsymbol{Z}_{j_\mathrm{L}})$ and $\boldsymbol{s}(\boldsymbol{Y}_n \boldsymbol{Z}_{j_1} \ldots \boldsymbol{Z}_{j_\mathrm{L}})$ for each $L = 1, \ldots, e_\mathrm{Z}$ and for each possible combination of $j_1, \ldots, j_\mathrm{L} \neq n$;
for $\ell = 1$ **to** $n-1$ **do**
    goodPick = 0;
    while *goodPick == 0* **do**
        Pick a random syndrome for $\boldsymbol{s}(\boldsymbol{X}_\ell)$ with weight in $\left[2\tilde{t}, n - 2\tilde{t} - 1\right]$ and $s_\ell\left(\boldsymbol{X}_\ell\right) = 0$;
        if $\boldsymbol{s}(\boldsymbol{Y}_\ell)$, $\boldsymbol{s}(\boldsymbol{X}_\ell \boldsymbol{Z}_{j_1} \ldots \boldsymbol{Z}_{j_\mathrm{L}})$ *and* $\boldsymbol{s}(\boldsymbol{Y}_\ell \boldsymbol{Z}_{j_1} \ldots \boldsymbol{Z}_{j_\mathrm{L}})$ *are not already assigned for all possible combinations* **then**
            goodPick = 1;
            Assign $\boldsymbol{s}(\boldsymbol{Y}_\ell)$ and all $\boldsymbol{s}(\boldsymbol{X}_\ell \boldsymbol{Z}_{j_1} \ldots \boldsymbol{Z}_{j_\mathrm{L}})$, $\boldsymbol{s}(\boldsymbol{Y}_\ell \boldsymbol{Z}_{j_1} \ldots \boldsymbol{Z}_{j_\mathrm{L}})$;
        end
        if *No more possible syndromes* **then**
            Restart the algorithm;
        end
    end
end
Construct the Stabilizer Matrix from $\boldsymbol{s}(\boldsymbol{X}_i)$ and $\boldsymbol{s}(\boldsymbol{Z}_i)$;
Check if all of the generators commute with each other.

Algorithm 1: Construction by syndrome assignment, $k = 1, e_\mathrm{g} = 1$.

5 Performance Analysis

It is well known that the *Codeword Error Rate* (CWER) for a standard $[[n, k]]$ QECC which corrects up to t generic errors per codeword is

$$P_\mathrm{e} = 1 - \sum_{j=0}^{t} \binom{n}{j} (1-\rho)^{n-j} \rho^j \tag{8}$$

where $\rho = p_\mathrm{x} + p_\mathrm{y} + p_\mathrm{z}$ is the error probability.

We now generalize this expression to an $[[n, k]]$ QECC which corrects up to e_g generic errors and up to e_Z Pauli $\boldsymbol{Z}$ errors per codeword. By weighting each pattern of correctable errors with the corresponding probability of occurrence, it is not difficult to show that the performance in terms of CWER is

$$P_\mathrm{e} = 1 - \sum_{j=0}^{e_\mathrm{g}+e_\mathrm{Z}} \binom{n}{j} (1-\rho)^{n-j} \xi(j; e_\mathrm{g}) \tag{9}$$

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where

$$\xi(j;e_\mathrm{g}) = \begin{cases} \rho^j & \text{if } j \le e_\mathrm{g} \\ \displaystyle\sum_{i=j-e_\mathrm{g}}^{j} \binom{j}{i} p_\mathrm{z}^i \sum_{\ell=0}^{j-i} \binom{j-i}{\ell} p_\mathrm{x}^\ell p_\mathrm{y}^{j-i-\ell} & \text{otherwise}\,. \end{cases} \tag{10}$$

In the case of asymmetric channels with $p_\mathrm{x} = p_\mathrm{y} = \rho/(A+2)$, $p_\mathrm{z} = A\rho/(A+2)$, and $A = p_\mathrm{z}/p_\mathrm{x}$ [36], the expression in (9) can be simplified to

$$P_\mathrm{e} = 1 - \sum_{j=0}^{e_\mathrm{g}+e_\mathrm{Z}} \binom{n}{j} (1-\rho)^{n-j} \rho^j \left(1 - 2^{j+1} \frac{(A/2)^{j-e_\mathrm{g}} - 1}{(A-2)(A+2)^j} u_{\mathrm{j}-e_\mathrm{g}-1}\right) \tag{11}$$

where $u_\mathrm{i} = 1$ if $i \ge 0$, otherwise $u_\mathrm{i} = 0$.

Using the previous expressions, we report in Fig. 1 the performance in terms of CWER for different codes, assuming an asymmetric channel. The parameter A accounts for the asymmetry of the channel, and for $A = 1$ we have the standard depolarizing channel. In the figure we plot the CWER for the new asymmetric $[[9,1]]$ code specified in Table 4 with $e_\mathrm{g} = 1$ and $e_\mathrm{Z} = 1$, over channels with asymmetry parameter $A = 1, 3$ and 10. For comparison, in the same figure we report the CWER for the known 5-qubits code, the Shor's 9-qubits code, both correcting $t = 1$ generic errors, and a $[[11,1]]$ code with $t = 2$ [37].

First, we note that for the symmetric codes the performance does not depend on the asymmetry parameter A, but just on the overall error probability ρ. For these codes, for a given t the best CWER is obtained with the shortest code. As expected, the performance of the new asymmetric $[[9,1]]$ code improves as A increases. In particular, for the symmetric channel, $A = 1$, the 5-qubits code performs better than the new one, due to its shorter codeword size. However, already with a small channel asymmetry, $A = 3$, the new code performs better than the 5-qubits code. For $A = 10$ the new code performs similarly to the $[[11,1]]$ symmetric code with $t = 2$. Asymptotically for large A, the channel errors tend to be of type $\boldsymbol{Z}$ only, and consequently the new code behaves like a code with $t = 2$.

6 Conclusions

We have investigated a new class of stabilizer short codes for quantum asymmetric Pauli channels, capable to correct up to e_g generic errors plus e_Z errors of type $\boldsymbol{Z}$. We generalized the quantum Hamming bound and derived the analytical expressions for the performance for the new codes. Then, we designed a $[[9,1]]$ QECC capable to correct up to 1 generic error plus 1 $\boldsymbol{Z}$ error, which is the shortest according to the new bound. The comparisons with known symmetric QECCs confirm the advantage of the proposed code in the presence of channel asymmetry.

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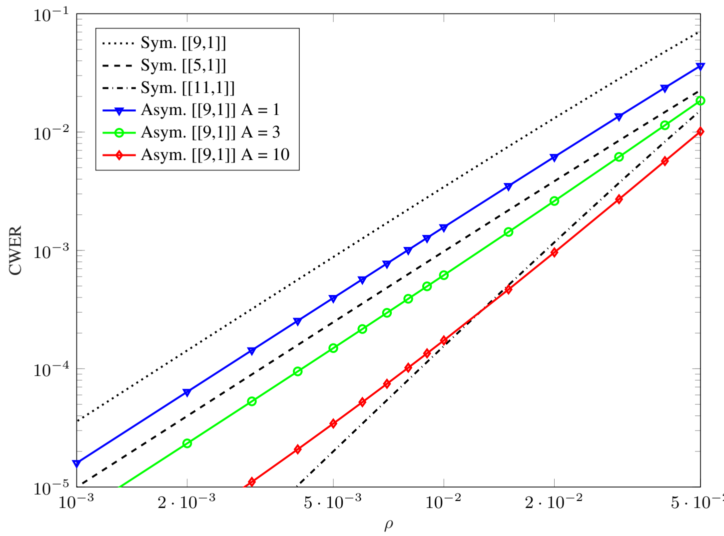

Fig. 1: Performance of short codes over an asymmetric channel, $k = 1$. Symmetric codes: 9-qubits code and 5-qubits code with $t = 1$, 11-qubits code with $t = 2$. Asymmetric 9-qubits code with $e_{\rm g} = 1, e_{\rm Z} = 1$.

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