Optimal representation of quantum channels

Paulina Lewandowska, Ryszard Kukulski, and Łukasz Pawela

Institute of Theoretical and Applied Informatics, Polish Academy of Sciences,
ul. Bałtycka 5, 44-100 Gliwice, Poland
plewandowska@iitis.pl
https://iitis.pl/en/person/plewandowska

Abstract. This work shows an approach to reduce the dimensionality of matrix representations of quantum channels. It is achieved by finding a base of the cone of positive semidefinite matrices which represent quantum channels. Next, this is implemented in the Julia programming language as a part of the QuantumInformation.jl package.

Keywords: Julia programming language, computational quantum information, quantum channels, convex cones, base of Hermiticity preserving maps

1 Introduction

Nowadays the fields of quantum information processing and machine learning are coming together leading to the emergence of quantum machine learning [1,2]. This area can be broadly divided into three, depending whether the data, algorithms or both are of quantum or classical nature. In this work we are interested in the case of quantum data being processed by a classical algorithm. The natural question arises: how this data should be represented and loaded into our algorithm? To be more precise, we are interested how to represent quantum channels in a succinct manner so that it can be an input into a classical neural network.

The goal of such a network would be to approximate, up to a reasonable error the distance between two channels Φ and Ψ. As Φ and Ψ are linear mappings transforming matrices into matrices it may not seem obvious how to define the distance between them. Turns out, there exists one notion of distance between channels which has an operational interpretation. The distance between Φ and Ψ can be expressed in the terms of so called diamond norm

$$\|\Phi - \Psi\|_\diamond = \max_{\|X\|_1=1} \| \left((\Phi - \Psi) \otimes \mathbb{1}\right)(X)\|_1. \tag{1}$$

This quantity plays a central role in the problem of quantum operation discrimination which has gained a lot of traction recently. This is due to the fact that this distance provides an upper bound on the probability of discrimination of Φ and Ψ.

ICCS Camera Ready Version 2020
To cite this paper please use the final published version:
DOI: 10.1007/978-3-030-50433-5_47

Consider a following setup. We are given a black box which is said to contain, with equal probability, either Φ or Ψ. What is the probability of guessing which of these is in the box if we are allowed to use the box only once? Turns out that this probability p is connected with the distance between Φ and Ψ [3]

$$p = \frac{1}{2} + \frac{1}{4}\|\Phi - \Psi\|_\diamond. \tag{2}$$

However the explicit form of the diamond norm contains an optimization over all input matrices X. In principle this can be solved via semidefinite programming, but regrettably this quickly becomes intractable with the growing dimension of the input matrix. That is why it would desirable to have the possibility to train a classical algorithm, like a neural network, on a relatively small set of quantum channels and have the ability to quickly approximate the distance between arbitrary channels utilizing this network.

That is why this paper aims at finding an optimal representation of quantum channels for the purposes of machine learning. By *optimal* we understand the lowest possible number of real parameters needed to define a quantum channel [4]. Further, we would like this representation to be technically usable so that we could train, for instance, neural networks to approximate functions of this objects. This approach could provide a large speed boost in the problem of quantum channel discrimination [3,5].

Our work is naturally divided into three parts. In the first part we show the mathematical structures needed to find the optimal representation. This involves dealing with cones of positive semidefinite matrices. The second part we present the example of whereas the last part presents the implementation of this example in the `Julia` language. This implementation is now a part of the `QuantumInformation.jl` [6,7] numerical library available on-line at `https://github.com/iitis/QuantumInformation.jl`. Surprisingly, despite the complex mathematical structure and quite technical proofs, the implementation is relatively simple and therefore useful.

2 Mathematical framework

2.1 Quantum channels

Let $\mathcal{X}, \mathcal{Y}$ be complex finite-dimensional vector spaces, let $\mathrm{L}(\mathcal{X}, \mathcal{Y})$ be the set of all linear operators transforming vectors from $\mathcal{X}$ to $\mathcal{Y}$ and denote $\mathrm{L}(\mathcal{X}) \coloneqq \mathrm{L}(\mathcal{X}, \mathcal{X})$. Further, consider mappings of the form

$$\Phi : \mathrm{L}(\mathcal{X}) \to \mathrm{L}(\mathcal{Y}). \tag{3}$$

The set of all such mappings will be denoted $\mathrm{T}(\mathcal{X}, \mathcal{Y})$ and $\mathrm{T}(\mathcal{X}) \coloneqq \mathrm{T}(\mathcal{X}, \mathcal{X})$. Quantum channels are such $\Phi \in \mathrm{T}(\mathcal{X}, \mathcal{Y})$ which are trace preserving and completely positive. The former means that

$$\forall A \in \mathrm{L}(\mathcal{X}) \quad \mathrm{Tr}(\Phi(A)) = \mathrm{Tr}(A). \tag{4}$$

ICCS Camera Ready Version 2020
To cite this paper please use the final published version:
DOI: 10.1007/978-3-030-50433-5_47

The latter is a bit more complicated. Formally this condition can be written as

$$\forall \mathcal{Z} \;\; \forall A \in \mathrm{L}(\mathcal{X} \otimes \mathcal{Z}) \;\; A \geq 0 \implies \left(\Phi \otimes \mathbb{1}_{\mathrm{L}(\mathcal{Z})}\right)(A) \geq 0. \tag{5}$$

The intuitive explanation is as follows. First, consider a $\rho \in \mathrm{L}(\mathcal{X})$ such that $\mathrm{Tr}(\rho) = 1$ and $\rho \geq 0$. Such an operator is called a quantum state. We would like our channels not only to transform states into states, but also we would like the ability to perform a channel on only a part of the system. In other words we would like the output of $\left(\Phi \otimes \mathbb{1}_{\mathrm{L}(\mathcal{Z})}\right)(\rho)$ to also be a proper quantum state for an arbitrary space $\mathcal{Z}$ and all $\rho \in \mathrm{L}(\mathcal{X} \otimes \mathcal{Z})$. This can only be fulfilled when we introduce the need for completely positivity. We will denote the set of all quantum channels as $\mathrm{C}(\mathcal{X}, \mathcal{Y})$ and $\mathrm{C}(\mathcal{X}) = \mathrm{C}(\mathcal{X}, \mathcal{X})$.

The mappings $\mathrm{T}(\mathcal{X}, \mathcal{Y})$ may be represented in a number of ways. For our purposes only the Choi-Jamiołkowski isomorphism [8,9] will be relevant. This representation states that there exists a bijection J between the sets $\mathrm{T}(\mathcal{X}, \mathcal{Y})$ and $\mathrm{L}(\mathcal{Y} \otimes \mathcal{X})$. This bijection can be explicitly written as

$$J(\Phi) = \sum_{i,j}^{\dim(\mathcal{X})} \Phi(|i\rangle\langle j|) \otimes |i\rangle\langle j| \,. \tag{6}$$

Φ is completely positive if and only if $J(\Phi) \geq 0$; Φ is trace preserving if and only if $\mathrm{Tr}_{\mathcal{Y}} J(\Phi) = \mathbb{1}_{\mathcal{X}}$. Finally, Φ is Hermiticity preserving if and only if $J(\Phi) \in \mathrm{Herm}(\mathcal{Y} \otimes \mathcal{X})$, where $\mathrm{Herm}(\mathcal{X})$ denotes the set of all Hermitian matrices in $\mathrm{L}(\mathcal{X})$.

2.2 Convex cone structures

Consider $\mathcal{X}$ is a real finite-dimensional vector space and $\mathcal{C} \subset \mathcal{X}$ is a closed convex cone. We assume that $\mathcal{C}$ is pointed, i.e. $\mathcal{C} \cap -\mathcal{C} = \{0\}$ and generating, i.e. for each $x \in \mathcal{X}$ there exists $u, w \in \mathcal{C}$ such that $x = u - w$. Such a cone $\mathcal{C}$ is called a proper cone in the space $\mathcal{X}$. The proper cone $\mathcal{C}$ becomes a partially ordered vector space $x \geq y \iff x - y \in \mathcal{C}$ for each $x, y \in \mathcal{X}$. Let $\mathcal{X}^*$ be the space dual to $\mathcal{X}$ defined by the inner product $\langle \cdot \mid \cdot \rangle$. Then, we may introduce a partial order in $\mathcal{X}^*$ as well with the dual cone

$$\mathcal{C}^* = \{f \in \mathcal{X}^* : \langle f \mid z \rangle \geq 0, \forall z \in \mathcal{C}\}. \tag{7}$$

The cone $\mathcal{C}^*$ is also closed and convex cone. If $\mathcal{C}$ is generating in space $\mathcal{X}$, then $\mathcal{C}^*$ is pointed and we may introduce partial order in $\mathcal{X}^*$ given by

$$f \geq g \iff f - g \in \mathcal{C}^* \tag{8}$$

for all $f, g \in \mathcal{X}^*$.

An interior point $e \in \mathrm{int}(\mathcal{C})$ of a cone $\mathcal{C}$ is called an order unit [10] if for each $x \in \mathcal{X}$, there exists $\lambda > 0$ such that $\lambda e - x \in \mathcal{C}$ whereas a base of $\mathcal{C}$ is defined as compact and convex subset $B \subset \mathcal{C}$ such that for every $z \in \mathcal{C} \setminus \{0\}$, there exists unique $t > 0$ and an element $b \in B$ such that $z = tb$. The following theorem shows there exists relation between the order unit e and a base of cone $\mathcal{C}$.

ICCS Camera Ready Version 2020
To cite this paper please use the final published version:
DOI: 10.1007/978-3-030-50433-5_47

Theorem 1. *The set $B_e = \{z \in \mathcal{C} : \langle e \mid z\rangle = 1\}$ is the base of $\mathcal{C}$ (determined by element e) if and only if an element e is an order unit and $e \in \operatorname{int}(\mathcal{C}^*)$.*

The proof of this theorem is presented in Appendix A.

2.3 Base of Hermiticity preserving maps

Let us now define the finite-dimensional linear space

$$\{\Phi \in \mathrm{T}(\mathcal{X}, \mathcal{Y}) : \Phi - \text{Hermiticity preserving}\}. \tag{9}$$

Due to the Choi–Jamiolkowski isomorphism, the set of all Hermiticity preserving linear maps of a finite-dimensional space is mathematically closely related to the set

$$\mathcal{V} = \{J(\Phi) : J(\Phi) \in \mathrm{Herm}(\mathcal{Y} \otimes \mathcal{X})\}, \tag{10}$$

of all Choi matrices of Hermiticity preserving maps.

In every linear space of Hermitian matrices $\mathrm{Herm}(\mathcal{Z})$ we can introduce an orthonormal basis $\mathcal{B}(\mathcal{Z})$. The basis $\mathcal{B}(\mathcal{Z})$ is a collection of $\dim(\mathcal{Z})^2$ matrices. The standard orthonormal basis is denoted by the set

$$\begin{aligned}\mathcal{B}(\mathcal{Z}) = &\\ &\left\{\frac{\mathbb{1}_{\mathcal{Z}}}{\sqrt{\dim(\mathcal{Z})}},\right.\\ &\frac{\sum_{a=1}^{k} |a\rangle\langle a| - k\,|k+1\rangle\langle k+1|}{\sqrt{k+k^2}}, \text{ for } k = 1, \ldots, \dim(\mathcal{Z}) - 1,\\ &\left.\frac{|a\rangle\langle b| + |b\rangle\langle a|}{\sqrt{2}}, \frac{i\,|a\rangle\langle b| - i\,|b\rangle\langle a|}{\sqrt{2}}, \text{ for } a, b = 1, \ldots, \dim(\mathcal{Z}) \text{ and } a \neq b\right\}.\end{aligned} \tag{11}$$

If we consider the space $\mathcal{V}$ of all Choi matrices of Hermiticity preserving maps we receive the $\dim(\mathcal{X})^2 \dim(\mathcal{Y})^2$ dimensional space. To reduce the number of dimensions of $\mathcal{V}$ we introduce the concept of a cone in this space and the base of cone.

Now we introduce a proper cone in the space $\mathcal{V}$ as

$$\mathcal{C} = \{J(\Phi) \in \mathcal{V} : J(\Phi) \geq 0\}, \tag{12}$$

and a subspace $\mathcal{S} \subset \mathcal{V}$ such that

$$\mathcal{S} = \{J(\Phi) \in \mathcal{V} : \mathrm{Tr}_{\mathcal{Y}} J(\Phi) = c\mathbb{1}_{\mathcal{X}}\,,\ c \in \mathbb{R}\}. \tag{13}$$

Fact 1 *The set of Choi matrices of quantum channels $\mathrm{C}(\mathcal{X}, \mathcal{Y})$ is the intersection of sets*

$$\mathcal{C} \cap \{J(\Phi) \in \mathcal{V} : \mathrm{Tr}_{\mathcal{Y}} J(\Phi) = \mathbb{1}_{\mathcal{X}}\}\,. \tag{14}$$

ICCS Camera Ready Version 2020
To cite this paper please use the final published version:
DOI: 10.1007/978-3-030-50433-5_47

We can also introduce the orthogonal complement $\mathcal{S}^\perp$ of $\mathcal{S}$ which is given by

$$\mathcal{S}^\perp := \{X \in \mathcal{V} : \mathrm{Tr}\,(XY) = 0, Y \in \mathcal{S}\}. \tag{15}$$

Fact 2 *The set $\mathcal{S}^\perp$ is given by*

$$\mathcal{S}^\perp = \{\mathbb{1}_\mathcal{Y} \otimes H : H \in \mathrm{Herm}(\mathcal{X}), \mathrm{Tr}(H) = 0\}. \tag{16}$$

The proof of this fact is presented in Appendix B.

We can also consider a proper cone $\mathcal{C}_\mathcal{S}$ in space $\mathcal{S}$ given by $\mathcal{C}_\mathcal{S} = \mathcal{S} \cap \mathcal{C}$ and a base $B_\mathcal{S} \subset \mathcal{C}_\mathcal{S}$ of the cone $\mathcal{C}_\mathcal{S}$. We can prove, using Theorem 1, that the set $B_\mathcal{S}$ is the base of cone $\mathcal{C}_\mathcal{S}$ if and only if $B_\mathcal{S} = \mathcal{S} \cap B_E$ for some order unit $E \in \mathrm{int}(\mathcal{C}^*)$. The base $B_\mathcal{S}$ determined by an order unit E will be denoted as $B_\mathcal{S}^E$ and is given by

$$B_\mathcal{S}^E = \{X \in \mathcal{C}_\mathcal{S} : \langle X \mid E \rangle = 1\}. \tag{17}$$

One can easily see that identity matrix $\mathbb{1}_\mathcal{Y} \otimes \mathbb{1}_\mathcal{X}$ is an order unit in cone $\mathcal{C}$. Thus we have the following observation.

Fact 3 *For $E := \frac{\mathbb{1}_\mathcal{Y} \otimes \mathbb{1}_\mathcal{X}}{\dim(\mathcal{X})}$ the base $B_\mathcal{S}^E$ is determined by the set of Choi matrices of quantum channels $\Phi \in \mathrm{C}(\mathcal{X}, \mathcal{Y})$ i.e.*

$$B_\mathcal{S}^E = \{J(\Phi) : \Phi \in \mathrm{C}(\mathcal{X}, \mathcal{Y})\}. \tag{18}$$

We are ready to establish the main result of our work.

Theorem 2. *The linear space $\mathcal{S}$ is the smallest linear subspace containing the set of quantum channels $\mathrm{C}(\mathcal{X}, \mathcal{Y})$ with orthonormal basis $\mathcal{B}(\mathcal{S})$ given by*

$$\left\{ \frac{\mathbb{1}_\mathcal{Y} \otimes \mathbb{1}_\mathcal{X}}{\sqrt{\dim(\mathcal{X})\dim(\mathcal{Y})}} \right\} \cup \left\{ G \otimes H : G \in \mathcal{B}(\mathcal{Y}) \setminus \left\{ \frac{\mathbb{1}_\mathcal{Y}}{\sqrt{\dim(\mathcal{Y})}} \right\}, H \in \mathcal{B}(\mathcal{X}) \right\}. \tag{19}$$

Moreover,

$$\dim(\mathcal{S}) = \dim(\mathcal{X})^2 \dim(\mathcal{Y})^2 - \dim(\mathcal{X})^2 + 1. \tag{20}$$

The proof of this theorem is presented in Appendix C.

Combining Theorem 2 with Fact 1 we obtain the following corollary.

Corollary 1. *Every quantum channel $\Phi \in \mathrm{C}(\mathcal{X}, \mathcal{Y})$ can be uniquely determined by $\dim(\mathcal{X})^2 \dim(\mathcal{Y})^2 - \dim(\mathcal{X})^2$ real numbers.*

Moreover, there exists extra, single non-zero coefficient which is fixed for all quantum channels $\mathrm{C}(\mathcal{X}, \mathcal{Y})$. Existence of this coefficient is a consequence of trace preserving condition $\mathrm{Tr}_\mathcal{Y} J(\Phi) = \mathbb{1}_\mathcal{X}$ and it can be calculated via

$$\left\langle \frac{\mathbb{1}_\mathcal{Y} \otimes \mathbb{1}_\mathcal{X}}{\sqrt{\dim(\mathcal{X})\dim(\mathcal{Y})}} \,\middle|\, J(\Phi) \right\rangle = \sqrt{\frac{\dim(\mathcal{X})}{\dim(\mathcal{Y})}}. \tag{21}$$

As a conclusion, we reduced the dimension of computational space by $\dim(\mathcal{X})^2$,

ICCS Camera Ready Version 2020
To cite this paper please use the final published version:
DOI: 10.1007/978-3-030-50433-5_47

3 Example

In this section we present how one can use the `Julia` language and `QuantumInformation.jl` library in order express quantum channels as vectors in the space $\mathcal{S}$.

Let us consider $\mathcal{X} = \mathbb{C}^2$ and $\mathcal{Y} = \mathbb{C}^3$ along with quantum channels $\Phi \in \mathrm{C}(\mathcal{X})$ given by

$$\Phi(X) = \frac{1}{2}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix} X \begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}, \quad X \in \mathrm{L}(\mathcal{X}), \tag{22}$$

and $\Psi \in \mathrm{C}(\mathcal{Y})$ defined as

$$\Psi(Y) = \begin{bmatrix}1 & 0.92-0.14i & 0.84-0.19i\\ 0.92+0.14i & 1 & 0.81+0.06i\\ 0.84+0.19i & 0.81-0.06i & 1\end{bmatrix} \odot Y, \quad Y \in \mathrm{L}(\mathcal{Y}), \tag{23}$$

where $\odot$ denotes the Hadamard product.

First we calculate the Choi matrices of Φ given by

$$J(\Phi) = \begin{bmatrix}0.5 & 0.5 & 0.5 & -0.5\\ 0.5 & 0.5 & 0.5 & -0.5\\ 0.5 & 0.5 & 0.5 & -0.5\\ -0.5 & -0.5 & -0.5 & 0.5\end{bmatrix}. \tag{24}$$

Analogously for Ψ we have

$$J(\Psi) = \begin{bmatrix}1 & 0\,0\,0 & 0.92-0.14i & 0\,0\,0 & 0.84-0.19i\\ 0 & 0\,0\,0 & 0 & 0\,0\,0 & 0\\ 0 & 0\,0\,0 & 0 & 0\,0\,0 & 0\\ 0 & 0\,0\,0 & 0 & 0\,0\,0 & 0\\ 0.92+0.14i & 0\,0\,0 & 1 & 0\,0\,0 & 0.81+0.06i\\ 0 & 0\,0\,0 & 0 & 0\,0\,0 & 0\\ 0 & 0\,0\,0 & 0 & 0\,0\,0 & 0\\ 0 & 0\,0\,0 & 0 & 0\,0\,0 & 0\\ 0.84+0.19i & 0\,0\,0 & 0.81-0.06i & 0\,0\,0 & 1\end{bmatrix}. \tag{25}$$

Now we use the function `channelbasis`. The inputs of this function are the dimensions of spaces $\mathcal{X}$ and $\mathcal{Y}$ of channels Φ, Ψ. The function returns an orthonormal basis of $\mathcal{S}$. Then, we are able to use the function `represent` which

ICCS Camera Ready Version 2020
To cite this paper please use the final published version:
DOI: 10.1007/978-3-030-50433-5_47

factor out Choi matrices $J(\Phi), J(\Psi)$ on basis elements and returns a vector representations $v_{J(\Phi)}, v_{J(\Psi)}$ of basis coefficients. In our examples we have

$$v_{J(\Phi)} = \begin{bmatrix} 1.0 \\ 0.70711 \\ 0.70711 \\ -0.70711 \\ -0.70711 \\ 1.0 \end{bmatrix} \oplus \mathbf{0}_7, \quad v_{J(\Psi)} = \begin{bmatrix} 0.70711 \\ -0.70711 \\ -0.20356 \\ -0.26978 \\ 1.29553 \\ 0.40825 \\ 0.40825 \\ -0.8165 \\ 0.08119 \\ 1.18231 \\ 1.14206 \\ 1.0 \end{bmatrix} \oplus \mathbf{0}_{61} \tag{26}$$

where $\mathbf{0}_i$ denotes vector of zeros of length i. If we want to reverse vector representation process, we can use function `combine`. The output matrix elements shall be accurate with original Choi matrix elements to 10^{-16} or better.

The explicit code of implementation in Julia language is presented in Appendix D.

4 Conclusion

In this work we find a matrix basis for quantum channels and provide strict mathematical proofs supporting our result. This basis allows us to reduce the dimensionality of the matrix which represents a quantum channel. This, in turn, allows us to speed up computation of a class of functions of these channels, which is applicable in, for instance, the study of quantum channel discrimination. Our analytical results are accompanied by functions written in the `Julia` language which decompose a given quantum channel in our basis. This implementation is now a part of the `QuantumInformation.jl` package [6,7].

Acknowledgements

This work was supported by the Foundation for Polish Science (FNP) under grant number POIR.04.04.00-00-17C1/18-00.

References

1. J. Biamonte, P. Wittek, N. Pancotti, P. Rebentrost, N. Wiebe, and S. Lloyd, "Quantum machine learning," *Nature*, vol. 549, pp. 195âĂŞ202, 2017.
2. P. Wittek, "Quantum machine learning: what quantum computing means to data mining," *Academic Press*, 2014.
3. C. W. Helstrom, "Quantum detection and estimation theory," *Academic press*, 1976.

ICCS Camera Ready Version 2020
To cite this paper please use the final published version:
DOI: 10.1007/978-3-030-50433-5_47

4. J. A. Holbrook, D. W. Kribs, and R. Laflamme, "Noiseless subsystems and the structure of the commutant in quantum error correction," *Quantum Information Processing*, vol. 2, no. 5, pp. 381–419, 2003.
5. A. Jenčová, "Base norms and discrimination of generalized quantum channels," *Journal of Mathematical Physics*, vol. 55, no. 2, p. 022201, 2014.
6. "QuantumInformation.jl;. Available from: `https:// github.com/iitis/QuantumInformation.jl`."
7. P. Gawron, D. Kurzyk, and Ł. Pawela, "QuantumInformation.jl—a julia package for numerical computation in quantum information theory," *PLoS ONE*, vol. 13, no. 12, p. e0209358, 2018.
8. M.-D. Choi, "Completely positive linear maps on complex matrices," *Linear Algebra and its Applications*, vol. 10, no. 3, pp. 285–290, 1975.
9. A. Jamiołkowski, "Linear transformations which preserve trace and positive semidefiniteness of operators," *Reports on Mathematical Physics*, vol. 3, no. 4, pp. 275–278, 1972.
10. B. Fuchssteiner and W. Lusky, "Convex cones," *Elsevier*, 2011.
11. W. Rudin, "Principles of Mathematical Analysis," *McGraw-hill New York*, vol. 3, 1964.
12. C. D. Meyer,"Matrix Analysis and Applied Linear Algebra,"*Siam*, vol. 71, 2000.

A Proof of Theorem 1

Proof. " $\Longrightarrow$ " Consider that B_e is a base of $\mathcal{C}$. An element $e \in \operatorname{int}(\mathcal{C}^*)$ if and only if there exists $r > 0$ such that the ball $K(e,r) \subset \mathcal{C}^*$, which is equivalent above condition

$$\exists_{r>0} \forall_{f \in \mathcal{X}^*} \left(||f - e|| < r \implies \forall_{z \in \mathcal{C}} \langle f \mid z \rangle \geq 0 \right). \tag{27}$$

By using the fact that in finite-dimensional spaces all norms are equivalent, we use the definition of induced norm given by

$$||f - e|| = \inf \left\{ M \in [0, \infty) : \forall_{x \in \mathcal{X}} | \langle f \mid x \rangle - \langle e \mid x \rangle | \leq M ||x|| \right\}. \tag{28}$$

Then, we have

$$||f - e|| < r \iff \exists_{0<M<r} \forall_{x \in \mathcal{X}} | \langle f \mid x \rangle - \langle e \mid x \rangle | \leq M ||x||. \tag{29}$$

Assume that $f \in \ K(e,r)$, $M \coloneqq \max\{||b|| : b \in B_e\}$ and $0 < r < \frac{1}{M}$. Then, we have

$$| \langle f \mid b \rangle - \langle e \mid b \rangle | \leq ||f - e|| \cdot ||b|| \leq r||b|| \leq rM < 1. \tag{30}$$

If $b \in B_e$, then $\langle e \mid b \rangle = 1$. Hence $| \langle f \mid b \rangle - 1| < 1$. That entails that $\langle f \mid b \rangle > 0$. By using the assumption we have $\langle f \mid z \rangle = t \langle f \mid b \rangle$, which implies that $\langle f \mid z \rangle > 0$.

" $\Longleftarrow$ " Now consider that $e \in \operatorname{int}(\mathcal{C}^*)$ is an order unit. It easy to see that B_e is a convex set. First we prove that $\langle e \mid z \rangle \neq 0$. Let $z \in \mathcal{C} \setminus \{0\}$ and $e \in \operatorname{int}(\mathcal{C}^*)$. If $e \in \mathcal{C}^*$, then $\langle e \mid z \rangle \geq 0$. It suffices to show that $\langle e \mid z \rangle \neq 0$. We will show this fact

ICCS Camera Ready Version 2020
To cite this paper please use the final published version:
DOI: 10.1007/978-3-030-50433-5_47

by contradiction. Assume that $\langle e \mid z\rangle = 0$ and let $\epsilon > 0$. By the Hahn–Banach theorem [11], there exists $z^* \in \mathcal{X}^*$ such that $\langle z^* \mid z\rangle = ||z||$. Then,

$$\langle e - \epsilon z^* \mid z\rangle = -\epsilon||z|| < 0 \tag{31}$$

It implies that $K(e,\epsilon) \notin \mathcal{C}^*$, which is contradiction with the assumption $e \in \operatorname{int}(\mathcal{C}^*)$. Therefore, $\langle e \mid z\rangle > 0$.

Let us see that if $b \coloneqq \frac{z}{\langle e \mid z\rangle}$ and $t \coloneqq \langle e \mid z\rangle$, then each element $z \in \mathcal{C} \setminus \{0\}$ can be written as $z = tb$. To prove that B_e is compact we note that $\mathcal{X}$ is a finite-dimensional space. Then the set B_e is compact if and only if B_e is closed and bounded. To prove that B_e is closed, we take any sequence $(z_n)_{n\in\mathbb{N}} \in B_e$ such that $z_n \xrightarrow{n\to\infty} z$. By the inner product continuity, we get

$$1 = \lim_{n\to\infty} \langle e \mid z_n\rangle = \left\langle e \,\middle|\, \lim_{n\to\infty} z_n \right\rangle = \langle e \mid z\rangle\,. \tag{32}$$

It implies that $z \in B_e$. Therefore B_e is closed. To prove that B_e is bounded we show there exists $M \in [0,\infty)$ such that $||z|| \le M$ for every $z \in B_e$. Let us take a compact sphere $\mathrm{S}(0,1)$ and closed cone $\mathcal{C}$. Then $S = \mathrm{S}(0,1) \cap \mathcal{C}$ is also compact. Notice the function $f : S \to \mathbb{R}_+$ given by $f(x) = \langle e \mid x\rangle$, where e is an order unit. By the Weierstrass theorem, a function f attains infimum and supremum. Therefore there exists $x_0 \in S$ such that $0 \le f(x_0) = \inf_{x\in S} f(x)$. Consider by contradiction that $f(x_0) = \langle e \mid x_0\rangle = 0$. We have $0 = \langle e \mid x_0\rangle = \langle e \mid tb_0\rangle = t$, where $b_0 \in B_e$, which is a contradiction with the assumption $t > 0$. Thus there exists $\lambda \coloneqq \langle e \mid x_0\rangle > 0$ such that $\langle e \mid z\rangle \ge \lambda||z||$ for every $z \in B_e$, hence $||z|| \le \frac{1}{\lambda}$. Taking $M \coloneqq \frac{1}{\lambda}$, we get thesis.

B Proof of Fact 2

Proof. It is clear that $\dim(\mathcal{V}) = (\dim(\mathcal{X})\dim(\mathcal{Y}))^2$. Consider a linear space $\mathcal{V} \oplus \mathbb{R}$ which is $(\dim(\mathcal{X})\dim(\mathcal{Y}))^2 + 1$ dimensional. Let $J(\Phi) \in \mathcal{S}$. The condition $\operatorname{Tr}_{\mathcal{Y}} J(\Phi) = c\mathbb{1}_{\mathcal{X}}, c \in \mathbb{R}$ in the space $\mathcal{V} \oplus \mathbb{R}$ is equivalent to

$$\begin{aligned}
&\sum_{k=1}^{\dim(\mathcal{Y})} \Re\left(J(\Phi)_{j+(k-1)\dim(\mathcal{X}),i+(k-1)\dim(\mathcal{X})}\right) = 0 \quad i > j,\\
&\sum_{k=1}^{\dim(\mathcal{Y})} \Im\left(J(\Phi)_{j+(k-1)\dim(\mathcal{X}),i+(k-1)\dim(\mathcal{X})}\right) = 0 \quad i < j,\\
&\sum_{k=1}^{\dim(\mathcal{Y})} J(\Phi)_{j+(k-1)\dim(\mathcal{X}),i+(k-1)\dim(\mathcal{X})} - c = 0 \quad i = j,
\end{aligned} \tag{33}$$

for all $i,j \in \{1,\dots,\dim(\mathcal{X})\}$. This homogeneous system of $\dim(\mathcal{X})^2$ linear equations is linearly independent. By rank–nullity theorem [12], we have

$$\dim(\mathcal{S}) = (\dim(\mathcal{X})\dim(\mathcal{Y}))^2 + 1 - \dim(\mathcal{X})^2. \tag{34}$$

Therefore, $\dim(\mathcal{S}^\perp) = \dim(\mathcal{X})^2 - 1$. To complete the proof, note that

$$\dim\left(\{\mathbb{1}_{\mathcal{Y}} \otimes H : H \in \operatorname{Herm}(\mathcal{X}), \operatorname{Tr}(H) = 0\}\right) = \dim(\mathcal{X})^2 - 1. \tag{35}$$

ICCS Camera Ready Version 2020
To cite this paper please use the final published version:
DOI: 10.1007/978-3-030-50433-5_47

C Proof of Theorem 2

Proof. According to Fact 3 the set $\mathrm{C}(\mathcal{X},\mathcal{Y})$ is the base of a proper cone $\mathcal{C}_{\mathcal{S}}$. That means

$$\operatorname{span}\left(\{J(\Phi)\in\mathcal{V} : J(\Phi)\geq 0, \operatorname{Tr}_{\mathcal{Y}} J(\Phi)=\mathbb{1}_{\mathcal{X}}\}\right)=\mathcal{S}. \tag{36}$$

Now we fix an orthonormal basis of the space $\mathcal{V}$. Let it be given as the collection

$$\mathcal{B}(\mathcal{V})=\{G\otimes H : G\in\mathcal{B}(\mathcal{Y}), H\in\mathcal{B}(\mathcal{X})\}. \tag{37}$$

By using Fact 2, if we take $X\in S^{\perp}$, than there exists $H\in\mathrm{Herm}(\mathcal{X})$, $\mathrm{Tr}(H)=0$ such that $X=\mathbb{1}_{\mathcal{Y}}\otimes H$. Let us set up the basis of $S^{\perp}$

$$\mathcal{B}(\mathcal{S}^{\perp})=\left\{\frac{\mathbb{1}_{\mathcal{Y}}}{\sqrt{\dim(\mathcal{Y})}}\otimes H : H\in\mathcal{B}(\mathcal{X})\backslash\left\{\frac{\mathbb{1}_{\mathcal{X}}}{\sqrt{\dim(\mathcal{X})}}\right\}\right\}. \tag{38}$$

Bearing in mind the relation $\mathcal{V}=\mathcal{S}\oplus\mathcal{S}^{\perp}$, we conclude that basis of $\mathcal{S}$ can be chosen as $\mathcal{B}(\mathcal{S})=\mathcal{B}(\mathcal{V})\backslash\mathcal{B}(\mathcal{S}^{\perp})$, namely

$$\left\{\frac{\mathbb{1}_{\mathcal{Y}}\otimes\mathbb{1}_{\mathcal{X}}}{\sqrt{\dim(\mathcal{X})\dim(\mathcal{Y})}}\right\}\cup\left\{G\otimes H : G\in\mathcal{B}(\mathcal{Y})\backslash\left\{\frac{\mathbb{1}_{\mathcal{Y}}}{\sqrt{\dim(\mathcal{Y})}}\right\}, H\in\mathcal{B}(\mathcal{X})\right\}, \tag{39}$$

which completes the proof.

D Julia implementation

Here, we present the code structure for the basis representation of Choi matrix of a qubit unitary channel Φ given by Eq. (22).

```
julia> using QuantumInformation

julia> H=hadamard(2)
2×2 Array{Float64,2}:
0.707107   0.707107
0.707107  -0.707107

julia> # defining Choi Matrix
J_Φ=res(H)*res(H)'
4×4 Array{Float64,2}:
0.5   0.5   0.5  -0.5
0.5   0.5   0.5  -0.5
0.5   0.5   0.5  -0.5
-0.5  -0.5  -0.5   0.5

julia> # representing Choi matrix in the basis of the subspace S
v_J_Φ=represent(channelbasis(Matrix{ComplexF64}, 2, 2),J_Φ)
13-element Array{Float64,1}:
0.0
0.9999999999999996
0.0
0.0
0.0
0.0
```

ICCS Camera Ready Version 2020
To cite this paper please use the final published version:
DOI: 10.1007/978-3-030-50433-5_47

```
0.0
0.0
0.7071067811865474
0.7071067811865474
-0.7071067811865474
-0.7071067811865474
0.9999999999999998

julia> # recovering the original Choi matrix from its basis representation
J_Φ_recovered=combine(channelbasis(Matrix{ComplexF64}, 2,2),v_J_Φ).matrix
4×4 Array{Complex{Float64},2}:
0.5+0.0im   0.5+0.0im   0.5+0.0im  -0.5+0.0im
0.5+0.0im   0.5+0.0im   0.5+0.0im  -0.5+0.0im
0.5+0.0im   0.5+0.0im   0.5+0.0im  -0.5+0.0im
-0.5+0.0im  -0.5+0.0im  -0.5+0.0im   0.5+0.0im

julia> # checking accuracy of recovery process using trace norm
print(norm_trace(J_Φ-J_Φ_recovered))
8.881784197001252e-16
```

ICCS Camera Ready Version 2020
To cite this paper please use the final published version:
DOI: 10.1007/978-3-030-50433-5_47