

Bézier surfaces for modeling inclusions in PIES

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Abstract. The paper presents the approach for solving 2D elastic boundary value problems defined in domains with inclusions with different material properties using the parametric integral equation system (PIES). The main feature of the proposed strategy is using Bézier surfaces for global modeling of inclusions. Polygonal inclusions are defined by bilinear surfaces, while others by bicubic surfaces. It is beneficial over other numerical methods (such as FEM and BEM) due to the lack of discretization. Integration over inclusions defined by surfaces is also performed globally without division into subareas. The considered problem is solved iteratively in order to simulate different material properties by applying initial stresses within the inclusion. This way of solving avoids increasing the number of unknowns and can also be used for elasto-plastic problems without significant changes. Some numerical tests are presented, in which the results obtained are compared with those calculated by other numerical methods.

Keywords: PIES · Inclusions · Bézier surfaces.

1 Introduction

Practical elastic problems very often require the analysis of piecewise homogeneous domains in which several regions exist, each with various material properties. Well known numerical methods such as the finite element method (FEM) [1–3] and the boundary element method (BEM) [1, 4–6] deal with such situations, however, they demand a completely different approach. Including two or more materials in FEM is quite straightforward, because different properties can be assigned to specific finite elements, which are always generated at the stage of body modeling, regardless of the problem solved. It is reduced to the proper discretization of the body and linking the right attributes with the right finite elements. On the other hand, most problems solved by BEM require defining only the boundary elements, which causes that elements inside the domain do not exist. Therefore, the only way to handle piecewise homogeneous problems in BEM is to divide the model into subregions or zones. Each zone has its own set of material properties and they are connected along a common interface. At the beginning, various zones are treated as separate BEM models, and finally they

are combined into a single system, using so-called constraint equations. The disadvantage of this strategy lies in the additional degrees of freedom arising from boundary elements on the common interface, which have to consider different results in separate zones.

The described above multi-region approach used in BEM results in the larger system of equations, therefore in [7, 8] the authors propose the different technique. The classical BEM is extended to include heterogeneous domains by introducing the volume effect. Such defined problem should be solved iteratively in order to modify the solution from the one with elastic homogeneous domain to this with presence of inclusions with different material properties. The only drawback of the proposed approach is that it requires cells for the evaluation of the domain integrals, which is an additional effort and is technically similar to discretization in FEM. Therefore, in order to overcome the need for a domain discretization, another concept is introduced in [9]. The new idea is to define subregion by two NURBS curves and a linear interpolation between them. It eliminates the need for cell generating, but the subdivision into integration regions still exists.

Taking into account mentioned above disadvantages, in this paper another approach is presented. The arising volume is not discretized into cells or defined by two curves, but is entirely modeled with one parametric surface. For subregions with linear boundary the surfaces of the first degree can be used, while for those with curvilinear edges the surfaces of the third degree are applied. Moreover, the evaluation of volume integrals is done in global manner, without dividing the area into subregions. Mentioned features (global modeling and integrating without classical discretization) are main advantages of the parametric integral equation system (PIES) [10]. This method was successfully used to solve 2D and 3D problems like potential [11], elastic [12], acoustic [13] and most recently elastoplastic [14]. As was emphasized above, PIES is characterized by no discretization and flexible way of modeling both the boundary and the domain by any curves and surfaces known from computer graphics [15, 16]. This crucial advantage comes from the fact that the shape is analytically included in the mathematical formalism of the method. Moreover, such approach gives a possibility for applying various methods for approximating of boundary and domain functions, because it is separated from approximating of the shape. Mentioned feature causes simple, independent improving of the accuracy of solutions without interfering with the shape of the boundary and the domain.

The main aim of the paper is to develop the approach for solving elastic problems with inclusions using PIES and global modeling of the shape. The idea bases on treating the whole solid as single region and simulating iteratively different material properties by applying initial stresses within the inclusion. The geometry of the inclusion is defined globally by Bézier surface. The approximation of initial stresses is performed by the Lagrange polynomial with various number and arrangement of interpolation nodes. Finally, the approach is verified compared to other well-known numerical methods.

2 Parametric integral equation system (PIES)

The parametric integral equation system (PIES) for elastic problems with inclusions taking into account incremental initial stress formulation can be presented in the following form

$$0.5\dot{\mathbf{u}}_l(\bar{s}) = \sum_{j=1}^n \int_{s_{j-1}}^{s_j} \{ \mathbf{U}_{lj}^*(\bar{s}, s) \dot{\mathbf{p}}_j(s) - \mathbf{P}_{lj}^*(\bar{s}, s) \dot{\mathbf{u}}_j(s) \} J_j(s) ds + \int_{\Omega} \mathbf{E}_l^*(\bar{s}, \mathbf{y}) \dot{\boldsymbol{\sigma}}_0(\mathbf{y}) d\Omega(\mathbf{y}), \quad (1)$$

where $\mathbf{U}_{lj}^*(\bar{s}, s)$, $\mathbf{P}_{lj}^*(\bar{s}, s)$, $\mathbf{E}_l^*(\bar{s}, \mathbf{y})$ are fundamental solutions for the displacements, tractions and strains respectively. Functions $\dot{\mathbf{p}}_j(s)$, $\dot{\mathbf{u}}_j(s)$ are incremental forms of parametric functions corresponding to tractions and displacements on the boundary. Vector $\dot{\boldsymbol{\sigma}}_0(\mathbf{y})$ contains increments of initial stresses inside the inclusion. s, \bar{s} are parameters in one-dimensional parametric reference system in which the boundary in PIES is defined and $s_{l-1} \leq \bar{s} \leq s_l$, $s_{j-1} \leq s \leq s_j$. In PIES s_{l-1} and s_{j-1} correspond to the beginning of l th and j th segments, while s_l and s_j to the end of these segments. $J_j(s)$ is the Jacobian, n is the number of boundary segments, $\mathbf{y} \in \Omega$ and $l, j = 1..n$.

The displacement fundamental solution $\mathbf{U}_{lj}^*(\bar{s}, s)$ is presented explicitly in [12] by

$$\mathbf{U}_{lj}^*(\bar{s}, s) = -\frac{1}{8\pi(1-\nu)\mu} \begin{bmatrix} (3-4\nu)\ln(\eta) - \frac{\eta_1^2}{\eta^2} & -\frac{\eta_1\eta_2}{\eta^2} \\ -\frac{\eta_1\eta_2}{\eta^2} & (3-4\nu)\ln(\eta) - \frac{\eta_2^2}{\eta^2} \end{bmatrix}, \quad (2)$$

where $\eta_1 = \Gamma_j^{(1)}(s) - \Gamma_l^{(1)}(\bar{s})$, $\eta_2 = \Gamma_j^{(2)}(s) - \Gamma_l^{(2)}(\bar{s})$, $\eta = [\eta_1^2 + \eta_2^2]^{0.5}$, ν is Poisson's ratio and μ is the shear modulus.

The traction fundamental solution $\mathbf{P}_{lj}^*(\bar{s}, s)$ is given by [12]

$$\mathbf{P}_{lj}^*(\bar{s}, s) = -\frac{1}{4\pi(1-\nu)\eta} \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}, \quad (3)$$

where

$$P_{11} = \left\{ (1-2\nu) + 2\frac{\eta_1^2}{\eta^2} \right\} \frac{\partial\eta}{\partial n}, \quad P_{22} = \left\{ (1-2\nu) + 2\frac{\eta_2^2}{\eta^2} \right\} \frac{\partial\eta}{\partial n},$$

$$P_{21} = P_{12} = \left\{ 2\frac{\eta_1\eta_2}{\eta^2} \frac{\partial\eta}{\partial n} - (1-2\nu) \left[\frac{\eta_1}{\eta} n_2(s) + \frac{\eta_2}{\eta} n_1(s) \right] \right\},$$

and $\frac{\partial\eta}{\partial n} = \frac{\partial\eta_1}{\partial n} n_1(s) + \frac{\partial\eta_2}{\partial n} n_2(s)$, while $n_1(s)$ and $n_2(s)$ are direction cosines of the external normal to j th segment of the boundary.

The strains fundamental solution $\mathbf{E}_l^*(\bar{s}, \mathbf{y})$ can be presented by

$$\mathbf{E}_l^*(\bar{s}, \mathbf{y}) = -\frac{1}{8\pi(1-\nu)G\bar{\eta}} \begin{bmatrix} 2A\bar{\eta}_1 - \bar{\eta}_1 + 2\bar{\eta}_1^3 & -\bar{\eta}_2 + 2\bar{\eta}_1^2\bar{\eta}_2 \\ A\bar{\eta}_2 + 2\bar{\eta}_1^2\bar{\eta}_2 & A\bar{\eta}_1 + 2\bar{\eta}_1\bar{\eta}_2^2 \\ A\bar{\eta}_2 + 2\bar{\eta}_1^2\bar{\eta}_2 & A\bar{\eta}_1 + 2\bar{\eta}_1\bar{\eta}_2^2 \\ -\bar{\eta}_1 + 2\bar{\eta}_1\bar{\eta}_2^2 & 2A\bar{\eta}_2 - \bar{\eta}_2 + 2\bar{\eta}_2^3 \end{bmatrix}^T, \quad (4)$$

where $A = (1 - 2\nu)$, $\bar{\eta} = [\bar{\eta}_1^2 + \bar{\eta}_2^2]^{0.5}$, $\bar{\eta}_1 = G^{(1)}(\mathbf{y}) - \Gamma_l^{(1)}(\bar{s})$ and $\bar{\eta}_2 = G^{(2)}(\mathbf{y}) - \Gamma_l^{(2)}(\bar{s})$.

The shape of the boundary and the domain is analytically included in (2),(3) and (4). The boundary can be modeled by means of any parametric curves $\mathbf{\Gamma}_j(s) = [\Gamma_j^{(1)}(s), \Gamma_j^{(2)}(s)]^T$ [15, 16], which are included into functions η_1 and η_2 . Functions $\bar{\eta}_1$ and $\bar{\eta}_2$ contain $\mathbf{G}(\mathbf{y}) = [G^{(1)}(\mathbf{y}), G^{(2)}(\mathbf{y}), G^{(3)}(\mathbf{y})]^T$, which is a parametric surface known from computer graphics [15, 16]. It should be emphasized that for 2D problems considered in this paper $G^{(3)}(\mathbf{y}) = 0$.

As can be seen formula (1) requires initial stresses inside the inclusion. They can be calculated using the strains and generalized Hooke's law. For this reason the integral identity for strains is also required.

3 Internal results

As mentioned in the previous section, the proposed strategy requires calculating strains inside the inclusion. They can be obtained using the following integral equation

$$\begin{aligned} \dot{\boldsymbol{\varepsilon}}(\mathbf{x}) = & \sum_{j=1}^n \int_{s_{j-1}}^{s_j} \left\{ \hat{\mathbf{D}}_j^*(\mathbf{x}, s) \dot{\mathbf{p}}_j(s) - \hat{\mathbf{S}}_j^*(\mathbf{x}, s) \dot{\mathbf{u}}_j(s) \right\} J_j(s) ds \\ & + \int_{\Omega} \hat{\mathbf{W}}^*(\mathbf{x}, \mathbf{y}) \dot{\boldsymbol{\sigma}}_0(\mathbf{y}) d\Omega(\mathbf{y}) + \hat{\mathbf{f}} \dot{\boldsymbol{\sigma}}_0(\mathbf{x}). \end{aligned} \quad (5)$$

The integrand $\hat{\mathbf{S}}_j^*(\mathbf{x}, s)$ is presented by the following formula (in plain strain)

$$\hat{\mathbf{S}}_j^*(\mathbf{x}, s) = \frac{1}{4\pi(1-\nu)r^2} \begin{bmatrix} S_{111} & S_{211} \\ S_{112} & S_{212} \\ S_{121} & S_{221} \\ S_{122} & S_{222} \end{bmatrix}, \quad (6)$$

$S_{111} = [2\frac{\partial r}{\partial n}[2\nu r_1 + r_1 - 4r_1^3] + (1 - 2\nu)(n_1 + 2r_1^2 n_1) + 4\nu r_1^2 n_1]$,
 $S_{112} = S_{121} = [2\frac{\partial r}{\partial n}[\nu r_2 - 4r_1^2 r_2] + (1 - 2\nu)(n_2 + 2r_1 r_2 n_1) + 2\nu(r_1^2 n_2 + r_1 r_2 n_1)]$,
 $S_{122} = [2\frac{\partial r}{\partial n}[r_1 - 4r_1 r_2^2] + (1 - 2\nu)(-n_1 + 2r_2^2 n_1) + 4\nu r_1 r_2 n_2]$,
 $S_{211} = [2\frac{\partial r}{\partial n}[r_2 - 4r_2 r_1^2] + (1 - 2\nu)(-n_2 + 2r_1^2 n_2) + 4\nu r_1 r_2 n_2]$,
 $S_{212} = S_{221} = [2\frac{\partial r}{\partial n}[\nu r_1 - 4r_2^2 r_1] + (1 - 2\nu)(n_1 + 2r_1 r_2 n_2) + 2\nu(r_2^2 n_1 + r_1 r_2 n_2)]$,
 $S_{222} = [2\frac{\partial r}{\partial n}[2\nu r_2 + r_2 - 4r_2^3] + (1 - 2\nu)(n_2 + 2r_2^2 n_2) + 4\nu r_2^2 n_2]$,
 where $\frac{\partial r}{\partial n} = \frac{\partial r_1}{\partial r} n_1(s) + \frac{\partial r_2}{\partial r} n_2(s)$, $r = [r_1^2 + r_2^2]^{0.5}$, $r_1 = \Gamma_j^{(1)}(s) - G^{(1)}(\mathbf{x})$ and $r_2 = \Gamma_j^{(2)}(s) - G^{(2)}(\mathbf{x})$.

The integrand $\hat{\mathbf{D}}_j^*(\mathbf{x}, s)$ can be described by (4) multiplied by -1, in which $\bar{\eta}$, $\bar{\eta}_1$, $\bar{\eta}_2$ are replaced by r, r_1, r_2 .

The last integral in (5) can be presented by the following expression

$$\hat{\mathbf{W}}^*(\mathbf{x}, \mathbf{y}) = \frac{1}{8\pi G(1-\nu)\bar{r}^2} \begin{bmatrix} W_{1111} & W_{1112} & W_{1121} & W_{1122} \\ W_{1211} & W_{1212} & W_{1221} & W_{1222} \\ W_{2111} & W_{2112} & W_{2121} & W_{2122} \\ W_{2211} & W_{2212} & W_{2221} & W_{2222} \end{bmatrix}, \quad (7)$$

$$\begin{aligned} W_{1111} &= [2(1-2\nu) - 1 + 8\nu\bar{r}_1^2 + 2(2\bar{r}_1^2 - 4\bar{r}_1^4)], \\ W_{1112} &= W_{1121} = W_{1211} = W_{2111} = [4\nu\bar{r}_1\bar{r}_2 + 2(\bar{r}_1\bar{r}_2 - 4\bar{r}_1^3\bar{r}_2)], \\ W_{1122} &= W_{2211} = [-1 + 2(\bar{r}_1^2 + \bar{r}_2^2 - 4\bar{r}_1^2\bar{r}_2^2)], \\ W_{1221} &= W_{1212} = W_{2112} = W_{2121} = [(1-2\nu) + 2\nu(\bar{r}_1^2 + \bar{r}_2^2) - 8\bar{r}_1^2\bar{r}_2^2], \\ W_{1222} &= W_{2122} = W_{2221} = W_{2212} = [4\nu\bar{r}_1\bar{r}_2 + 2(\bar{r}_1\bar{r}_2 - 4\bar{r}_1\bar{r}_2^3)], \\ W_{1111} &= [2(1-2\nu) - 1 + 8\nu\bar{r}_2^2 + 2(2\bar{r}_2^2 - 4\bar{r}_2^4)], \end{aligned}$$

where $\bar{r} = [\bar{r}_1^2 + \bar{r}_2^2]^{0.5}$, $\bar{r}_1 = G^{(1)}(\mathbf{y}) - G^{(1)}(\mathbf{x})$ and $\bar{r}_2 = G^{(2)}(\mathbf{y}) - G^{(2)}(\mathbf{x})$.

The free term from (5) for the plane strain case is given by

$$\hat{\mathbf{f}} = -\frac{1}{16G(1-\nu)} \begin{bmatrix} 1 - 2(3-4\nu) & 0 & 0 & -(3-4\nu) \\ 0 & -(3-4\nu) & 1 & 0 \\ 0 & 1 & -(3-4\nu) & 0 \\ -(3-4\nu) & 0 & 0 & 1 - 2(3-4\nu) \end{bmatrix}. \quad (8)$$

4 Solving PIES

Solving problems without inclusions by PIES is reduced to finding functions $\dot{\mathbf{u}}_j(s)$ and $\dot{\mathbf{p}}_j(s)$. They are approximated using series with various base functions, e.g. in this paper the Lagrange polynomials are used [14]. After substituting such series into PIES, writing down the resultant equation at all interpolation (collocation) points and reordering the following system of equation is obtained

$$\mathbf{A}\dot{\mathbf{x}} = \dot{\mathbf{b}}, \quad (9)$$

where \mathbf{A} is a matrix that contains mixed values of both integrands (2,3), $\dot{\mathbf{x}}$ contains the unknown boundary values, while $\dot{\mathbf{b}}$ contains prescribed values on the boundary.

If inclusions are considered, this equation should be extended by the additional term

$$\mathbf{A}\dot{\mathbf{x}} = \dot{\mathbf{b}} + \dot{\mathbf{f}}, \quad (10)$$

where $\dot{\mathbf{f}}$ includes integrals (4) with the initial stresses.

Initial stresses are also approximated using series similar to those applied for boundary functions, but they depend on two variables

$$\dot{\boldsymbol{\sigma}}_0(\mathbf{y}) = \sum_{r=0}^{R_1-1} \sum_{w=0}^{R_2-1} \dot{\boldsymbol{\sigma}}_0^{rw}(\mathbf{y}) L_{rw}(\mathbf{y}), \quad (11)$$

where

$$L_{rw}(\mathbf{y}) = L_r(y_1)L_w(y_2),$$

$$L_r(y_1) = \prod_{o=0, o \neq r}^{R_1} \frac{y_1 - y_{1o}}{y_{1r} - y_{1o}}, \quad L_w(y_2) = \prod_{o=0, o \neq w}^{R_2} \frac{y_2 - y_{2o}}{y_{2w} - y_{2o}},$$

and $N = R_1 \times R_2$ is the given number of interpolation nodes, while $\dot{\boldsymbol{\sigma}}_0^{rw}(\mathbf{y})$ is the value of initial stress at interpolation node with (y_1, y_2) coordinates.

As was mentioned in the previous section, to calculate initial stresses strains are required. The final matrix form of the formula for calculating strains can be obtained in a similar as above manner using (5)

$$\dot{\boldsymbol{\epsilon}} = -\mathbf{A}' \dot{\mathbf{x}} + \mathbf{b}' + (\mathbf{W} + \mathbf{F}) \dot{\boldsymbol{\sigma}}_0, \quad (12)$$

where $(\mathbf{W} + \mathbf{F})$ correspond to expressions (7,8), \mathbf{A}' is a matrix that contains mixed values of two first integrands in (5), while \mathbf{b}' contains prescribed values on the boundary.

Integrals required for (10) and most of integrals for (12) are regular or weakly singular, therefore they are calculated using Gauss integration and the subdivision technique applied to the surfaces [4–6]. The integral over the domain with function (7), which should be calculated in (12), is strongly singular, with singularity of order $\frac{1}{r^2}$ for 2D problems. The singularity is isolated by replacing the integral by two integrals. The first is weakly singular and is treated by the subdivision technique, while second is transformed into the boundary as presented in [17].

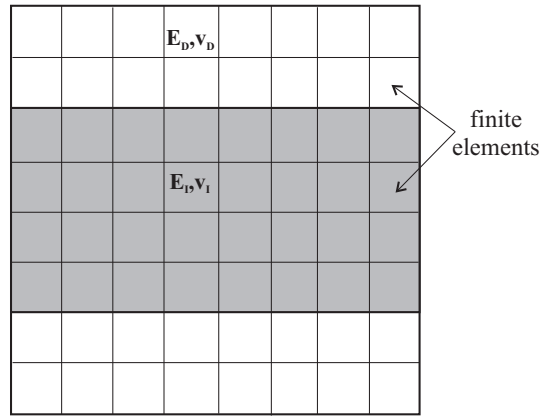
5 Modeling of inclusions

As can be seen in (1) and (5), last integrals require defining the domain of inclusion. In FEM, regardless of the problem, the whole body is modeled by finite elements. Therefore, considering the inclusion requires assuming various material properties for various groups of elements (Fig. 1).

In BEM, the inclusion is divided into so-called cells, which technically resemble finite elements (Fig. 2). In both methods the number, type and arrangement of those elements influence also the accuracy of the solutions. Such approach is troublesome and often forces the use of more elements than the shape of the geometry actually requires. Moreover, integration also is performed over elements, even if mapping method is used [9], which bases on modeling the inclusion using two NURBS curves and interpolation between them.

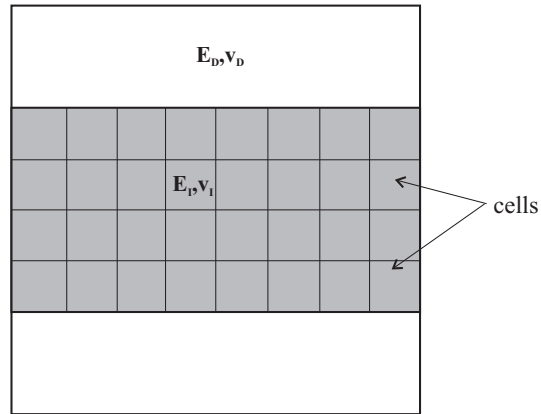
For this reason in this paper another approach is proposed. It bases on the popular tool of computer graphics, namely on surface patches [15, 16]. Unlike other methods, the inclusion area is modeled entirely using a single Bézier surface, but also other types of surfaces can be used instead. If the polygonal inclusion is considered, then bilinear surfaces should be used (Fig. 3), while for other curvilinear shapes of subdomains bicubic surfaces can be applied (Fig. 4).

As can be seen in Fig. 3 and Fig. 4, the proposed way of definition requires smaller number of points than in FEM and BEM. For example the inclusion presented in Fig. 3 is created using only 4 control points (■), which are actually



E_D, ν_D - material properties of the body
 E_I, ν_I - material properties of the inclusion

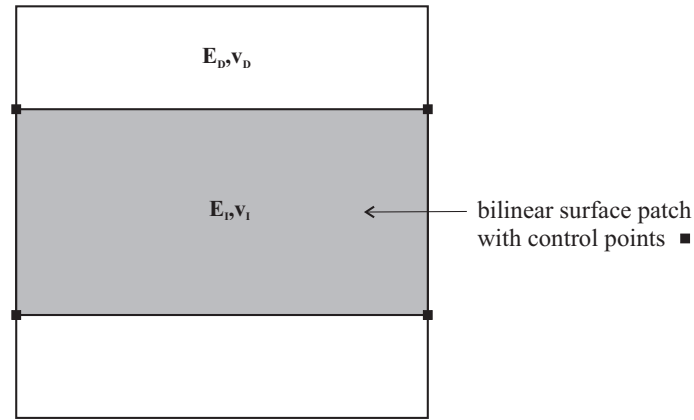
Fig. 1. Modeling the inclusion in FEM



E_D, ν_D - material properties of the body
 E_I, ν_I - material properties of the inclusion

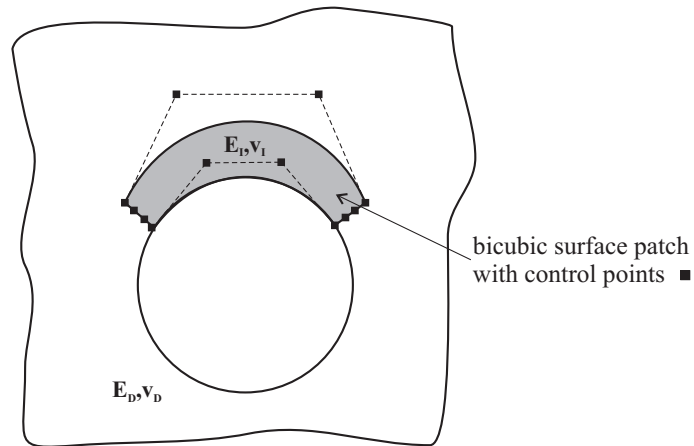
Fig. 2. Modeling the inclusion in BEM

corner points. For comparison, the same inclusion in FEM (Fig. 1) and BEM (Fig. 2) is composed of 32 finite elements and 32 cells respectively. If we assume that those elements and cells are linear, both cases require 128 nodes for defining them, while FEM additionally needs finite elements for modeling the whole body (64 elements and 256 nodes in total). The curvilinear shape of the inclusion in Fig. 4 requires 12 control points (■), which define its boundary (other 4 are not important due to 2D nature of the problem).



E_D, ν_D - material properties of the body
 E_I, ν_I - material properties of the inclusion

Fig. 3. Modeling the polygonal inclusion in PIES



E_D, ν_D - material properties of the body
 E_I, ν_I - material properties of the inclusion

Fig. 4. Modeling the curvilinear inclusion in PIES

Moreover, the proposed approach gives the opportunity for simple modification of the defined shape. Moving even one control point causes significant change in the shape of the inclusion. Such feature can be very useful when dealing with identification or optimization of the shape. Furthermore, modification of the geometry automatically modifies the mathematical formalism of PIES (1), because the shape is included in it. It also allows for separation of shape model-

ing from the approximation of solutions, which results in possibility of applying various methods for both stages of solving boundary problems.

6 Iterative procedure

The general iterative procedure can be adapted to various kinds of inclusions (elastic, inelastic) and various formulations (initial strains, initial stress). In this paper only elastic inclusions are considered, while the problem is formulated as initial stress. The following steps have to be performed in order to approximate final results [7–9]:

- a) The elastic problem is solved assuming that there is no inclusions (using (9))
 $\mathbf{A}\dot{\mathbf{x}}_{i=0} = \dot{\mathbf{b}}$.
- b) The strains $\dot{\boldsymbol{\varepsilon}}$ are calculated within the inclusion (using (12)).
- c) The total boundary and internal results are initialized
 $\mathbf{x}_{total} = \dot{\mathbf{x}}_{i=0}$,
 $\boldsymbol{\varepsilon}_{total} = \dot{\boldsymbol{\varepsilon}}_{i=0}$.
- d) Convert the internal strains into the initial stresses using
 $\dot{\boldsymbol{\sigma}}_0 = (C_D - C_I)\dot{\boldsymbol{\varepsilon}}$,
 where C_D is the constitutive matrix of the domain used in a) as a homogeneous, while C_I is the constitutive matrix for the inclusion.
- e) Compute the last integral over the domain from (1) using kernel (4) and obtained in d) values of $\dot{\boldsymbol{\sigma}}_0$. The result of that operation is the residual vector $\dot{\mathbf{f}}_i$.
- f) Check if the residual vector is sufficiently small. If yes the iterative process ends, otherwise it continues.
- g) The residual vector obtained in e) is applied as the right hand side for the system of equation (9)
 $\mathbf{A}\dot{\mathbf{x}}_i = \dot{\mathbf{f}}_i$.
- h) The above system of equations is solved and once again the strains inside the inclusions are calculated.
- i) The final boundary and internal results are updated
 $\mathbf{x}_i = \mathbf{x}_{i-1} + \dot{\mathbf{x}}_i$,
 $\boldsymbol{\varepsilon}_i = \boldsymbol{\varepsilon}_{i-1} + \dot{\boldsymbol{\varepsilon}}_i$.
- j) Repeat the procedure from step e).

7 Numerical verification and discussion

The example concerns a square plate ($2m \times 2m$) with a square inclusion in the center ($1m \times 1m$). The considered body is fixed at the bottom and loaded on the top with a constant pressure $p = 1MN/m^2$ (Fig. 5). The plate is in plane strain conditions and it is composed of two different materials. The material of the plate (D) and the material of the inclusion (I) are characterized by the following properties: $E_D = 5000MN/m^2$, $\nu_D = 0.3$ and $E_I = 2500MN/m^2$, $\nu_I = 0.3$.

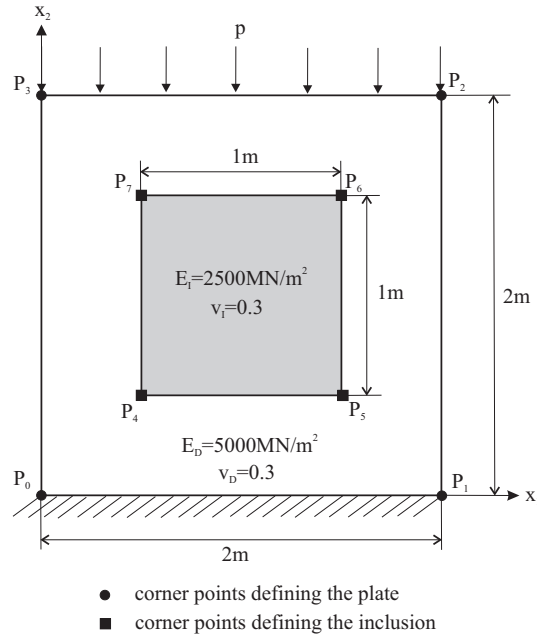


Fig. 5. Geometry and boundary conditions for the square plate with the inclusion

At first, the main feature of the proposed approach is analyzed - the way of modeling of inclusions. As can be seen in Fig. 5, the boundary of the plate in PIES is modeled by four linear Bézier segments, using two corner \bullet points for each of them. Such approach requires only four corner points (P_0, P_1, P_2, P_3). The inclusion is defined using single Bézier surface of the first degree, which is created also by only four corner points \blacksquare (P_4, P_5, P_6, P_7).

For comparison, the considered body was modeled using other numerical methods. The model corresponding to FEM was designed with the help of 400 quadratic finite elements (100 of them concern the inclusion itself). It means that the proposed approach allows modeling with significantly fewer input data than classical FEM (even several hundred times). The described above FEM model was also used for numerical calculations presented later in this section.

The way of modeling of inclusions in PIES was also compared with BEM. For this purpose, models available in the literature were used. Two approaches were taken into account: first concerns classical discretization into cells [7], while second presented in [9] uses NURBS curves and a linear interpolation between them. Using classical BEM approach the plate was defined by 48 quadratic boundary elements, while the inclusion by 36 quadratic cells. In the isogeometric BEM two linear NURBS curves are used. As can be seen, the first way (classical BEM) requires defining several times more nodes than in PIES, while the second uses only 4 control points to define two curves, but approximated area should still be divided into subareas for integration.

It should be emphasized that in classical versions of the so-called element methods (FEM, BEM), the number of elements, their type and arrangement (shape approximation) are closely related to the accuracy of the obtained solutions (the approximation of solutions). In many cases, the more elements, the higher the accuracy. However, it should be remembered that the more elements, the greater the system of equations to solve. In PIES the approximation of solutions is independent of the approximation of the shape. Therefore, only a minimal amount of data is used for shape modeling, while the accuracy is steered by the number and arrangement of interpolation points (e.g. see formula (11) for initial stresses).

Vertical displacements on the right half of the upper segment of the body were obtained. It took only a few iterations to obtain such results in PIES. They are based on the model presented in Fig. 5 with 20 interpolation nodes assumed for the approximation of the boundary results and 16 for the initial stresses. Obtained displacements were compared with those returned by FEM (using the model described above) and they both are presented in Fig. 6.

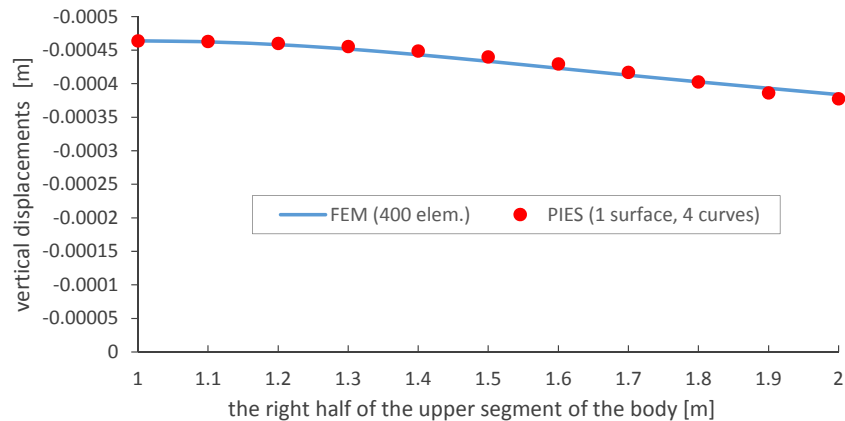


Fig. 6. Vertical displacements on the right half of the upper segment of the body obtained by FEM and PIES

In Fig. 6 very good agreement between FEM and PIES results can be noticed. It should be remembered that they were obtained with a completely different amount of data used for modeling and a completely different amount of solved equations (both numbers in favor of PIES).

The next stage of studies concerns comparison of the deformed shape of the boundary taking into account the body with and without the inclusion. Obtained by PIES boundaries are shown in Fig. 7 and Fig. 8. The way of deformation of

the body with the inclusion simulated by PIES agrees with that presented in [18] obtained by BEM.

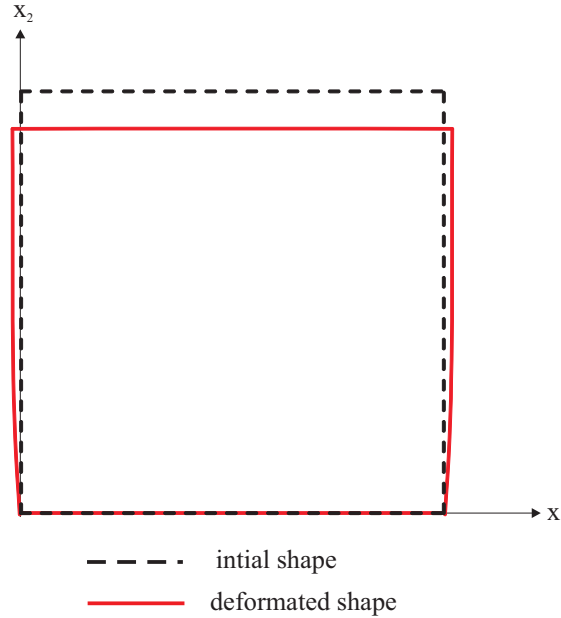


Fig. 7. The deformed boundary for the body without the inclusion

The presented in this section results confirm the high efficiency of the proposed method (efficient modeling of inclusions with the small number of data) and also its accuracy (the results are consistent with another methods). The effectiveness of the method measured by the calculation time in comparison with other methods was not checked. It comes from the fact that it is not reasonable to compare the time of execution of the program created by the authors for the research being considered with the commercial product for finite element analysis.

8 Conclusions

The paper presents the PIES method for solving 2D elastic boundary value problems with inclusions. The geometry of the inclusion is defined using Bézier surface, without classical discretization. Such approach reduces the number of data required for modeling and gives possibility for easy modification of the shape. PIES separates the approximation of the shape from the approximation

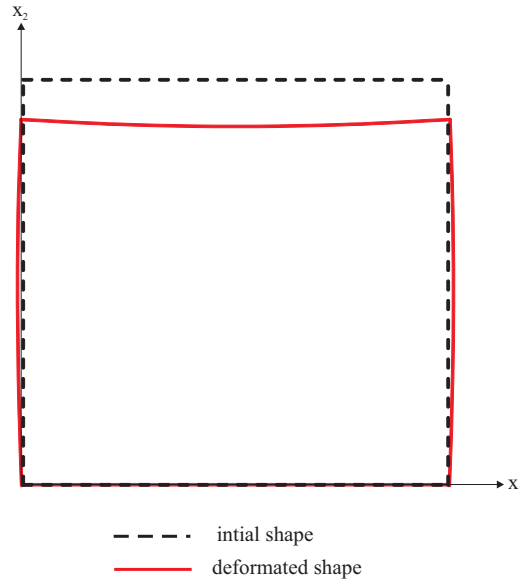


Fig. 8. The deformed boundary for the body with the inclusion

of the solutions, therefore for displacements, tractions and initial stresses approximation, Lagrange polynomials are used. The accuracy of solution in such case depends only on the number and arrangement of interpolation nodes.

The verification of the proposed approach was performed on the example with elastic inclusion in comparison to other numerical methods. Obtained results are in good agreement with FEM and BEM solutions. It should be mentioned that PIES is especially efficient taking into account the way of modeling of inclusions, but the accuracy is also satisfactory.

The proposed strategy requires tests on more complicated examples, especially when the inclusions with nonlinear material behavior are present.

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