Projection-Based Model Reduction Using Asymptotic Basis Functions

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Abstract. Galerkin projection provides a formal means to project a differential equation onto a set of preselected basis functions. This may be done for the purpose of formulating a numerical method, as in the case of spectral methods, or formulation of a reduced-order model (ROM) for a complex system. Here, a new method is proposed in which the basis functions used in the projection process are determined from an asymptotic (perturbation) analysis. These asymptotic basis functions (ABF) are obtained from the governing equation itself; therefore, they contain physical information about the system and its dependence on parameters contained within the mathematical formulation. This is referred to as reduced-physics modeling (RPM) as the basis functions are obtained from a physical model-driven, rather than data-driven, technique. This new approach is tailor-made for modeling multiscale problems as the various scales, whether overlapping or distinct in time or space, are formally accounted for in the ABF. Regular- and singular-perturbation problems are used to illustrate that projection of the governing equations onto the ABF allows for determination of accurate approximate solutions for values of the "small" parameter that are much larger than possible with the asymptotic expansion alone and naturally accommodate multiscale problems in which large gradients occur in adjacent regions of the domain.

Keywords: Galerkin projection · Asymptotic methods · Reduced-order modeling.

1 Introduction

Projection-based methods are frequently used as the basis for numerical methods, such as spectral methods, and formulation of reduced-order models (ROM), such as proper-orthogonal decomposition (POD). A ROM can then be used as a simplified mathematical model with a reduced number of degrees-of-freedom for systems involving complex physics in optimization, control, and system identification settings, for example. Projection methods approximate a solution in terms of a linear combination of preselected basis functions. In spectral methods, the basis functions are chosen for their ease of integration and other desirable mathematical properties. In ROM, the basis functions are computed from numerical or experimental data using an optimization procedure, such as POD, which is a "data-driven" method.

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The rise in popularity of spectral numerical methods and ROM techniques has highlighted the need for determining basis functions that are appropriate for the particular problem under consideration. A "model-driven" approach is proposed here in which the basis functions are obtained directly and formally from the governing equations rather than data obtained from the system. Tailor-made for such a purpose, particularly in multiscale problems, is asymptotic (or perturbation) methods, which constitute a set of techniques for obtaining an asymptotic series in terms of a physical parameter within the system that becomes very small or large. They provide a powerful set of tools that allow one to learn a great deal about a system directly from the governing equation(s) alone without the need to solve it, simulate it, or conduct an experiment. Such techniques lead to important physical insights that would be difficult to glean numerically or experimentally alone – a need that only becomes more acute as we seek to solve increasingly complex problems in multiscale physics, multidisciplinary design optimization, and control. The primary interest here is in applications to differential equations in which multiple spatial and/or temporal scales are present within the system.

2 Projection Methods

Projection methods, and the closely associated method of weighted residuals, have their origins in variational methods (Cassel 2013). Consider a general non-homogeneous differential equation of the form

$$\mathcal{L}u(x) = f(x), \tag{1}$$

where the differential operator $\mathcal{L}$ may be linear or nonlinear. In the *inverse problem*, the differential Euler equation (1) is converted into its *proper variational form*. To do so, the inner product of the differential equation (1) with the variation of the dependent variable is taken, i.e. the differential equation is projected onto δu according to

$$\int_{x_0}^{x_1} (\mathcal{L}u - f)\, \delta u \, dx = 0, \tag{2}$$

which is known as the *reduced variational form*. Carrying out the necessary integration by parts leads to the proper variational form if one exists.

In the Galerkin method, an approximation to the solution $u(x)$ is devised in the form of a linear combination of a set of basis functions of the form

$$\bar{u}(x) = \phi_0(x) + \sum_{n=1}^{N} c_n \phi_n(x) = \phi_0(x) + c_1\phi_1(x) + \cdots + c_n\phi_n(x) + \ldots + c_N\phi_N(x). \tag{3}$$

The linearly-independent *basis functions* $\phi_n(x)$, $n = 0, \ldots, N$ that comprise this *trial function* account for the spatial dependence in the solution and are specified functions that satisfy the boundary conditions. Any non-homogeneous boundary conditions are satisfied by $\phi_0(x)$, such that the remaining basis functions are homogeneous at the boundaries of the domain.

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Because the trial function $\bar{u}(x)$ is only an approximate solution, it does not satisfy the differential equation (1) exactly. As a proxy for the unknown error of this approximate solution, the *residual* is defined from the differential equation by

$$r(x) = \mathcal{L}\bar{u}(x) - f(x). \tag{4}$$

In general, the differential operator could be linear or nonlinear, steady or unsteady, and ordinary or partial; however, a one-dimensional framework will be used here in order to introduce the method. The reduced variational form (2) can be written in terms of the residual and trial function as

$$\int_{x_0}^{x_1} r(x)\delta\bar{u}\, dx = 0. \tag{5}$$

Although the reduced-variational form (2) is only useful as an exact representation of the so-called *weak form* of the differential equation, the analogous form (5) in terms of the trial function is simply setting the inner product of the residual with the variation of the trial function to zero, i.e. it is enforcing an orthogonal projection.

Let us more closely examine what equation (5) is indicating. Because the basis functions $\phi_n(x)$, $n = 0, \ldots, N$ are specified and do not vary, and it is only the coefficients c_n, $n = 1, \ldots, N$ that vary, taking the variation of the trial function (3) and substituting into equation (5) yields

$$\int_{x_0}^{x_1} r(x)\left[\phi_1(x)\delta c_1 + \cdots + \phi_n(x)\delta c_n + \ldots + \phi_N(x)\delta c_N\right] dx = 0.$$

Because the coefficients are arbitrary, for this sum to vanish, the expression multiplying each variation must vanish. That is,

$$\int_{x_0}^{x_1} r(x)\phi_i(x)\, dx = 0, \quad i = 1, \ldots, N. \tag{6}$$

The index is changed to i so that there is no confusion with the index n that identifies the basis functions in the residual $r(x)$. Note that each of the orthogonal projections (6) includes all coefficients c_n and basis functions $\phi_n(x)$, $n = 1, \ldots, N$ in the residual but only one of the basis functions $\phi_i(x)$. This is referred to as an orthogonal projection, not because the basis functions must be mutually orthogonal, but because orthogonality of the residual and basis functions is being enforced inherently in the method. Evaluating these N definite integrals removes the dependence on the spatial coordinate x and leads to an $N \times N$ system of algebraic equations for the coefficients c_n, $n = 1, \ldots, N$. If the problem is unsteady, then this process will lead to an $N \times N$ system of ordinary differential equations for the time-dependent coefficients $c_n(t)$, $n = 1, \ldots, N$. This system of algebraic or ordinary differential equations is the ROM.

The solution to the ROM produces the coefficients that for the given basis functions lead to the trial function that is closest to the exact stationary function $u(x)$. This is typically called the *Galerkin method* when applied as a numerical

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method, whereas it is referred to as *Galerkin projection* when applied in ROM. It is helpful, however, to realize that all such methods trace their roots back to the inverse variational problem. The Galerkin method is particularly straightforward when the basis functions are mutually orthogonal, in which case all of the products of basis functions $\phi_n(x)\phi_i(x)$ in equation (6) vanish except when $n = i$.

Within the Galerkin method, we may select the basis functions in two ways depending on our objective:

1. Preselect the basis functions for their ease of integration and orthogonality properties. This gives rise to *spectral numerical methods*, in which Fourier series or Chebyshev polynomials are typically used as the basis functions.
2. Calculate the basis functions from numerical or experimental data obtained from the system for a particular set of parameters. This is done by solving an optimization problem and gives rise to *proper orthogonal decomposition* (POD) and its extensions and is the basis of the ROM for the system's behavior.

For more on spectral methods, see Fletcher (1984) and Canuto et al. (1988), and for more on ROM, see Rowley and Dawson (2017).

3 Asymptotic (Perturbation) Methods

Clearly, the effectiveness of projection-based spectral numerical methods and ROM hinge on the choice of basis functions used in the trial function. This is where asymptotic methods may prove to be transformational. Once again, the ideal basis functions would contain as much information about the physics of the system as possible, thereby minimizing the number of modes required to obtain an accurate spectral method or ROM. While POD forms the basis functions from numerical or experimental data obtained from the solution itself for a given set of parameters, the *asymptotic basis functions* (ABF) to be put forward here contain the parametric dependence within them and thus apply over a wide parameter range. More to the point, POD is a data-driven method that does not take advantage of any knowledge of the system's mathematical model, whereas such a model is the basis for obtaining the ABF that are the centerpiece of the method introduced here.

Asymptotic (perturbation) methods are a collection of techniques, including matched asymptotic expansions, multiple scales, WKB theory, and strained coordinates for treating systems containing a small or large parameter. The analysis results in the so-called distinguished limit, gauge functions, and asymptotic series. The distinguished limit exposes the dominant balances of terms in the governing equation(s) in the limiting case and indicates the size of the various regions in a domain. The gauge functions in the small parameter allow us to quantify the level of approximation of each term in the asymptotic expansion. Finally, the terms in the asymptotic series provide increasingly higher-order approximations of the system.

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An asymptotic expansion is a parametric expansion in the small parameter, say ϵ, of the form [1]

$$u(x;\epsilon) = \sum_{n=1}^{\infty} g_n(\epsilon) u_n(x;\epsilon) = g_1(\epsilon)u_1(x;\epsilon) + g_2(\epsilon)u_2(x;\epsilon) + \cdots, \tag{7}$$

where $g_n(\epsilon)$ are the *gauge functions* and show the asymptotic orders of the successive terms. The accuracy of the expansion improves as ϵ is reduced and/or as additional terms are included in the asymptotic expansion. An asymptotic expansion is local in ϵ, i.e. it applies for $\epsilon \ll 1$, but it is global in x, i.e. it applies for all x in the domain. Often only a small number of terms are necessary in an asymptotic series for a good approximation of the overall solution. The $u_n(x;\epsilon)$ functions provide useful information about the dominant behavior of the system when the parameter ϵ is small, and only a small number of terms are typically required to decipher this information.

An asymptotic sequence of gauge functions $g_1(\epsilon), g_2(\epsilon), \ldots, g_n(\epsilon), \ldots$ is an asymptotic sequence as $\epsilon \to 0$ if

$$g_{n+1}(\epsilon) \ll g_n(\epsilon) \quad \text{as} \quad \epsilon \to 0^+, \tag{8}$$

or equivalently

$$\lim_{\epsilon \to 0^+} \frac{g_{n+1}(\epsilon)}{g_n(\epsilon)} = 0, \quad n = 1, 2, 3, \ldots. \tag{9}$$

The most common situation is when the gauge functions are simply integer powers of ϵ

$$g_1(\epsilon) = \epsilon^0 = 1, \quad g_2(\epsilon) = \epsilon, \quad g_3(\epsilon) = \epsilon^2, \quad \ldots,$$

which clearly satisfy the above properties of an asymptotic sequence. Therefore, an asymptotic sequence exhibits an asymptotic convergence rate in terms of the small parameter.

In general, the small parameter ϵ could be in the equation(s), boundary or initial conditions, and/or the domain geometry (e.g. thin-airfoil theory and thin-shell theory). In regular-perturbation problems, the small parameter does not multiply the highest-order derivative term(s) in the differential equation, and a single expansion is uniformly valid over the entire domain. In singular-perturbation problems, however, the small parameter multiplies the highest-order derivative term so that the order of the equation is reduced for $\epsilon = 0$. In this case, different expansions must be obtained in separate regions of the domain, each with its own dominant physics. The method of matched asymptotic expansions ensures that neighboring expansions formally match with one another. It is in this way that asymptotic analysis reveals the dominant physics

[1] Note that traditionally in asymptotic methods, the terms are numbered in the asymptotic expansions starting with zero. That is, the *leading-order term* is $g_0(\epsilon)u_0(x;\epsilon)$. Here, however, we start the expansion from unity in order to be consistent with our RPM nomenclature. In this way, $u_0(x)$ can be used to accommodate non-homogeneous boundary conditions as in spectral methods.

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within each region of the domain where there is fundamentally different, i.e. multiscale, behavior arising from the governing equation(s). That is, the effect of the small parameter is not "small" qualitatively. In particular, singular-perturbation problems are such that the solution with $\epsilon = 0$ is of a fundamentally different form and does not smoothly approach the leading-order solution as $\epsilon \to 0^+$.

A common criticism of asymptotic methods is that the resulting asymptotic expansions often only agree closely with the exact or numerical solution of the governing equation(s) for small values of the parameter ϵ. However, the contention here is that their use in projection methods holds great promise in extending the relevance of the asymptotic series to a wider parameter range for a given system or even to other similar systems. For more details on asymptotic methods, see Hinch (1991) and E (2011).

4 Asymptotic Basis Functions

Presenting these brief overviews of projection and asymptotic methods side-by-side leads one to ask the seemingly obvious question, "Why not use the terms in an asymptotic expansion as the basis functions for a ROM?" Would not these modes take into account the important physical parameter(s) in the system such that the same modes could be used over a wide range of such parameters? Recall that POD modes only apply for the specific value of the parameter(s) for which the data was obtained. New values of the parameters means new data, which means new POD modes. Also recall that spectral numerical methods have difficulties dealing with solutions having large gradients within the domain. What if the chosen basis functions in the trial function actually became more accurate, rather than less so, when large gradients appear in a solution and even fewer basis functions are required?

Because ABF are obtained from the physical model, we refer to them as *reduced-physics models* (RPM); not only does it allow for reduction in the *order* of the model for the system, as with ROM, it contains valuable physical information about the system. In some sense, it is the natural progression of basis functions used in projection methods:

1. Spectral numerical methods - choose Fourier, Legendre, or Chebyshev functions for computational efficiency and ease of integration in the projection process.
2. ROM, e.g. POD - use *data-driven* modes obtained directly from experimental or numerical data for the given problem.
3. RPM, e.g. ABF - use *model-driven* modes obtained directly from the governing equation(s) for the given problem (or a similar one).

The ABF could be used in a spectral method or ROM context. That is, they could be used in any setting involving the use of preselected basis functions for modeling a system having a small parameter. Although we lose the advantages of orthogonal basis functions in RPM, this is more than made up for by the extreme reduction in the number of required basis functions enabled by use of the ABF.

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In large part, this is facilitated by the fact that the dependence on the small parameter is accounted for in the gauge functions, which form an asymptotic sequence, and the asymptotic basis functions themselves, which is not the case in traditional ROM techniques.

The ABF must be uniformly valid across the entire domain in order to provide global basis functions in the projection process. This is naturally the case for regular-perturbation problems or when the method of multiple scales is used. For singular-perturbation problems using matched asymptotic expansions, on the other hand, the *composite solution* must be formed from the asymptotic series in each distinct region of the domain having their own distinguished limits in terms of the small parameter.

5 Regular-Perturbation Illustration

A regular-perturbation problem is considered first for two reasons. First, it will be shown that the RPM coefficients approach unity as $\epsilon \to 0$. This confirms that the Galerkin projection process is consistent with asymptotic series expansions and preserves the asymptotic solution, which is simply the RPM with $c_n = 1$ for $n = 1, \ldots, N$. Secondly, it will be shown that the ABF can be used with Galerkin projection in the RPM to obtain accurate solutions for values of ϵ that are much larger than is possible for the asymptotic solution alone.

Consider the ordinary differential equation

$$\frac{d^2u}{dx^2} + 2\epsilon\frac{du}{dx} + u = 1, \quad u(0) = 0, \quad u\left(\frac{\pi}{2}\right) = 0, \tag{10}$$

where $0 \leq \epsilon \ll 1$, i.e. ϵ is a small, but positive, parameter. Thus, we have a differential equation of the form

$$\mathcal{L}u(x;\epsilon) = f(x),$$

where

$$\mathcal{L}u(x;\epsilon) = u''(x) + 2\epsilon u'(x) + u(x), \quad f(x) = 1.$$

The ABF are obtained through a regular-perturbation analysis followed by computing the RPM coefficients by projecting the governing equation (10) onto the ABF.

5.1 Asymptotic Basis Functions

Because the small parameter does not multiply the highest-order derivative in the governing equation (10), a regular-perturbation expansion is expected to be suitable of the form

$$u(x;\epsilon) = u_1(x) + \epsilon u_2(x) + \epsilon^2 u_3(x) + \ldots, \tag{11}$$

where the gauge functions are integer powers of ϵ. Substituting the asymptotic expansion (11) into the differential equation (10) and equating like powers of

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ϵ results in a series of differential equations and boundary conditions for each order in the asymptotic series. Solving each of these equations in succession leads to the first three orders:

$$\begin{aligned} u_1(x) &= 1 - \sin x - \cos x, \\ u_2(x) &= x \sin x - \frac{\pi}{2} \sin x + x \cos x, \\ u_3(x) &= \frac{1}{8} \left[\left(2\pi(2x+1) - 4(x+1)x + \pi^2 \right) \sin x - 4(x-1)x \cos x \right]. \end{aligned} \tag{12}$$

Given these expressions, the first three terms in the regular-perturbation expansion (11) are

$$u(x;\epsilon) = u_1(x) + \epsilon u_2(x) + \epsilon^2 u_3(x) + O(\epsilon^3).$$

As in other ROM settings, the solution $u(x;\epsilon)$ of the differential equation is approximated in terms of a linear combination of a finite number of ABF according to:

$$u(x;\epsilon) = \sum_{n=1}^{N} c_n \phi_n(x;\epsilon) = c_1\phi_1(x;\epsilon) + \cdots + c_n\phi_n(x;\epsilon) + \ldots + c_N\phi_N(x;\epsilon). \tag{13}$$

The ABF are the product of each order of the asymptotic solution and their corresponding gauge functions, i.e. $\phi_n(x;\epsilon) = g_n(\epsilon)u_n(x;\epsilon)$, and the c_n, $n = 1, \ldots, N$ are the RPM coefficients to be determined from the projection process. Observe that the asymptotic expansion (7) is the sum of the asymptotic basis functions, which corresponds to the RPM solution (13) with all $c_n = 1$. Therefore, the RPM coefficients provide a quantitative measure of the accuracy of the asymptotic solution without knowledge of the exact or numerical solution. In this regular-perturbation case, the ABF are given by

$$\phi_1(x;\epsilon) = u_1(x), \quad \phi_2(x;\epsilon) = \epsilon u_2(x), \ldots, \quad \phi_n(x;\epsilon) = \epsilon^{n-1} u_n(x), \ldots. \tag{14}$$

Note that rather than the basis functions being characterized by modes with increasing frequencies of oscillation or higher-order polynomials as n increases, as in traditional spectral methods, the asymptotic basis functions form an asymptotic sequence according to equations (8) and (9). Moreover, adding additional terms, i.e. increasing N, does not influence the lower-order ABF already obtained; only the coefficients in the RPM need to be recomputed.

5.2 Reduced-Physics Model

Substituting the RPM expansion (13) for the solution $u(x;\epsilon)$, the spatial definite integrals in the projection equations (6) eliminate the explicit dependence on the spatial coordinate x and produce a system of N algebraic equations for the coefficients c_n, $n = 1, \ldots, N$ for a given value of ϵ. Observe that the continuous, infinite-dimensional ordinary differential equation $\mathcal{L}u = f$ has been converted into a system of N algebraic equations for the RPM coefficients in the trial function. This is the RPM.

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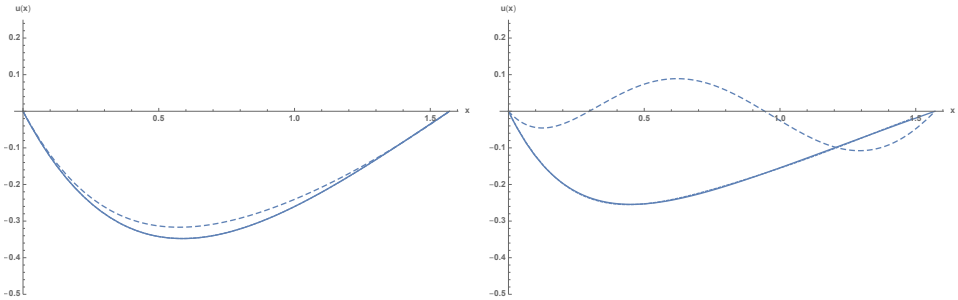
Table 1. RPM coefficients for the regular-perturbation problem (10) using Galerkin projection with $N = 4$ asymptotic basis functions; L_2-norm of error for asymptotic expansion (AE) and reduced-physics model (RPM) as compared to the exact solution.

ϵ	c_1	c_2	c_3	c_4	$\|e_{AE}\|_2$	$\|e_{RPM}\|_2$
0.01	1.000000	1.000000	0.999964	0.999963	3.14167×10^{-10}	3.0188×10^{-12}
0.1	0.999999	0.999999	0.996393	0.996393	3.13265×10^{-6}	3.00695×10^{-8}
1.0	0.992809	0.992809	0.728974	0.728974	2.45447×10^{-2}	2.12738×10^{-4}
2.0	0.939427	0.939427	0.383799	0.383799	0.249235	1.65804×10^{-3}
3.0	0.841983	0.841983	0.197772	0.197772	0.843277	4.04899×10^{-3}
4.0	0.728941	0.728941	0.107342	0.107342	1.94203	6.74403×10^{-3}
5.0	0.619753	0.619753	0.0616782	0.0616782	3.69399	9.38273×10^{-3}
10.0	0.273812	0.273812	0.00736198	0.00736198	27.8312	1.66313×10^{-2}



(a) $\epsilon = 1$. (b) $\epsilon = 2$.

Fig. 1. Exact (solid), asymptotic (dashed), and RPM (dotted) solutions for regular-perturbation problem with $N = 4$.

The results of the Galerkin projection for the regular-perturbation problem (10) with $N = 4$ ABF are given in Table 1 and Figure 1. Recall that the values of the RPM coefficients provide a quantitative measure of how accurate the asymptotic expansion is. The RPM coefficients resulting from the projection for various values of ϵ show that the RPM coefficients indeed approach unity as $\epsilon \to 0$ as expected. This confirms that the projection process is consistent with the asymptotic series. Observe from the table that the coefficients begin to deviate from being close to unity for $\epsilon > 0.1$, above which the RPM and asymptotic solutions deviate substantially. The deviation between the asymptotic and RPM solutions is illustrated in Figure 1(a) for $\epsilon = 1$. While the asymptotic solution displays a notable error compared to the exact solution, the RPM solution is indistinguishable from the exact solution.

As shown in Figure 1(b) for $\epsilon = 2$, the asymptotic solution does not even agree qualitatively with the exact solution (as expected), whereas the RPM solution is still graphically indistinguishable from the exact solution. Also given in Table 1 is the L_2-norm of the error between the asymptotic and RPM solutions

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as compared to the exact solution of the ordinary differential equation (10). Observe that the L_2-norm of the error for the RPM solution is two orders of magnitude smaller than that for the underlying asymptotic solution for $\epsilon < 3$. While the error for the asymptotic solution increases dramatically thereafter, that for the RPM increases very slowly up to $\epsilon = 10$. Consequently, as the value of ϵ increases, the asymptotic series is not at all representative of the actual solution. However, the RPM solution remains quite accurate with the L_2-norm only increasing marginally up to $\epsilon = 10$. This is well beyond the range of validity of the asymptotic solution alone, which is obtained in the limit $\epsilon \to 0$. With no additional information, therefore, the accuracy of the asymptotic solution can be improved dramatically for increasing values of ϵ using Galerkin projection with ABF!

6 Singular-Perturbation Illustration

The objective of the singular-perturbation problem considered in this section is to illustrate how the ABF can be used with Galerkin projection to obtain increasingly accurate solutions as $\epsilon \to 0$ with a small number of basis functions as compared to other projection-based methods, such as spectral methods. In particular, the ABF capture the multiscale behavior of the singular-perturbation problem for cases where traditional projection methods actually require a rapidly increasing number of terms to accurately resolve. In fact, because the ABF are the exact solution in the limit as $\epsilon \to 0$, fewer ABF are actually required. In addition, just as for the regular-perturbation problem, Galerkin projection using ABF allows for accurate solutions with larger values of the small parameter than possible with the asymptotic solution alone.

Consider the ordinary differential equation

$$\epsilon \frac{d^2u}{dx^2} + 2\frac{du}{dx} + u = 1, \quad u(0) = 0, \quad u(1) = 0, \tag{15}$$

where $0 \leq \epsilon \ll 1$. Observe that this is very similar to the regular-perturbation problem (10) except that the small parameter now multiplies the highest-order derivative in the governing equation (15); therefore, a *singular-perturbation expansion* is expected to be appropriate. In fact, there is an $O(\epsilon)$ (distinguished limit) thin boundary layer near the boundary at $x = 0$. The *method of matched asymptotic expansions* then leads to separate *outer* and *inner asymptotic expansions* for $x = O(1)$ and $x = O(\epsilon)$, respectively. Matching in the overlap region between the two regions leads to a uniformly-valid *composite solution* that applies across the entire domain. In this case, an asymptotic expansion is obtained of the form

$$u(x;\epsilon) = u_1(x) + \epsilon u_2(x) + \ldots. \tag{16}$$

Here, only the first two terms are considered, i.e. $N = 2$ in the ABF, in the Galerkin projection to obtain the RPM. Once again, the ABF include both the asymptotic solutions and their corresponding gauge functions.

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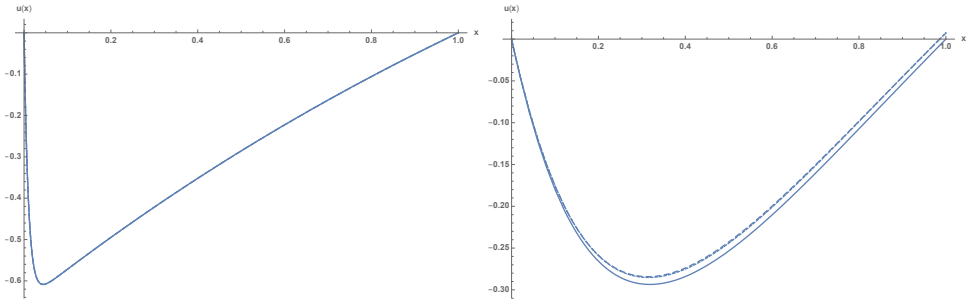
Table 2. RPM coefficients for the singular-perturbation problem (15) using Galerkin projection with $N = 2$ asymptotic basis functions; L_2-norm of error for asymptotic expansion (AE) and reduced-physics model (RPM) as compared to the exact solution; and the number of Chebyshev polynomials N_{Cheb} required to obtain the same level of accuracy as the RPM with $N = 2$.

ϵ	c_1	c_2	$\|e_{AE}\|_2$	$\|e_{RPM}\|_2$	N_{Cheb}
0.02	0.999985	1.01089	2.21793×10^{-5}	5.17765×10^{-6}	39
0.05	0.999911	1.02711	1.28291×10^{-4}	3.06292×10^{-5}	26
0.1	0.999663	1.05360	4.43086×10^{-4}	1.09102×10^{-4}	17
0.2	0.998849	1.10231	1.25002×10^{-3}	3.49802×10^{-4}	11
0.3	0.998031	1.13953	2.97762×10^{-3}	1.85862×10^{-3}	7
0.4	0.997889	1.15391	8.71716×10^{-3}	7.99755×10^{-3}	5
0.5	0.999162	1.03925	2.01347×10^{-2}	2.02258×10^{-2}	5



(a) $\epsilon = 0.02$. (b) $\epsilon = 0.4$.

Fig. 2. Exact (solid), asymptotic (dashed), and RPM (dotted) solutions for singular-perturbation problem with $N = 2$. Observe the thin boundary layer near $x = 0$.

The results for Galerkin projection with the $N = 2$ ABF for the singular-perturbation problem are shown in Table 2 and Figure 2. Once again, the values of the RPM coefficients approach unity as $\epsilon \to 0$. Although not as dramatic as for the regular-perturbation problem, the RPM solution with ABF is found to be more accurate than the asymptotic solution alone for values of the small parameter that are larger than would be possible otherwise. Unlike in the regular-perturbation case, the small parameter ϵ must remain sufficiently small to maintain the integrity of the asymptotic structure. That is, ϵ must remain sufficiently small such that the boundary-layer thickness remains smaller than the overall domain size. The right column of the table shows the number of Chebyshev polynomials required in a traditional spectral method to obtain the same level of accuracy as the RPM solution with only two ABF. Observe the dramatic increase in the number of Chebyshev polynomials required as the boundary layer gets thinner and the associated gradients get larger as $\epsilon \to 0$.

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7 Conclusions and Discussion

A novel approach to projection methods has been developed in which the terms in an asymptotic (perturbation) expansion are used as basis functions within a Galerkin projection framework. Using regular- and singular-perturbation problems based on ordinary differential equations, it is illustrated how 1) the projection method is consistent with the asymptotic expansion in the sense that as the small parameter goes to zero, the RPM coefficients approach unity, 2) the projection process allows a small number of ABF to form the basis for very accurate solutions for values of the "small" parameter that are well beyond those for the asymptotic series itself, and 3) Galerkin projection with only two ABF for the singular-perturbation problem result in increasingly accurate solutions for decreasing values of the small parameter, while the number of Chebyshev polynomials required to obtain an equally accurate solution increases dramatically as $\epsilon \to 0$.

The present paper illustrates the potential of this novel approach to dramatically reduce the computational requirements for solving multiscale problems having large local gradients in their solutions. In its simplest form, the RPM approach introduced here simply provides another, but more physical, source of basis functions for spectral and ROM-based techniques. The significance of this new method as it relates to asymptotic methods, spectral numerical methods, and reduced-order modeling are discussed in the remainder of the discussion.

7.1 From an Asymptotic Methods Perspective

The primary strength of asymptotic methods is their ability to illuminate the dominant physics in various regions of multiscale problems for small (or large) values of a physical parameter. Their primary weakness is that the range of validity of the resulting asymptotic expansion is generally very limited and not known unless a separate analytical or numerical solution is available. RPM with ABF addresses both of these issues. By projecting the governing equation(s) onto the ABF, the resulting RPM coefficients provide a quantitative measure of the range of validity of the asymptotic solution that is self-consistent and does not require any external means of verification – the closer the coefficients are to unity, the better the asymptotic expansion.

While determining the range of validity of an asymptotic expansion is primarily of academic interest, applying Galerkin projection with ABF to dramatically increase the range of validity of an asymptotic expansion to $O(1)$ values of the "small" parameter has a great deal of practical usefulness. Crucially, no additional information is required to do so. In addition to extending the validity of an asymptotic expansion, the physical insight inherent in the asymptotic method allows one to see how $\epsilon = O(1)$ cases relate to the limiting behavior to determine to what extent the dominant balances obtained as $\epsilon \to 0$ persist to larger ϵ. All of this comes from the original differential equation(s) governing the physics of the system without requiring any numerical solutions or experiments.

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Essentially, the ABF provide another source of basis functions for numerical methods, such as spectral methods, ROM, such as proper-orthogonal decomposition, system identification, such as SINDy, and other modeling tasks as discussed below. Rather than simply being chosen for their mathematical properties, such as in spectral methods, or being obtained from numerical or experimental data, such as in POD, however, the ABF are obtained from the governing equations themselves. Therefore, they provide a *model-driven*, rather than *data-driven*, approach to obtaining basis functions for further calculation or analysis. Consequently, an inherently physically-based multiscale method is derived with the characteristic scales being accommodated in the ABF.

7.2 From a Spectral Numerical Methods Perspective

Recall that spectral numerical methods are based on Galerkin projection (or another weighted residuals method) and utilize basis functions preselected for their ease of integration and other mathematical properties, such as orthogonality, for example. Typically, Fourier modes, Legendre polynomials, or Chebyshev polynomials are used. Spectral methods lead to very accurate global solutions with very fast "spectral" convergence rates. The latter is the case, however, only when the solution is sufficiently smooth. Solutions containing large gradients require a large number of spectral modes to accurately resolve the solution.

Singular-perturbation asymptotic expansions display the opposite behavior. They provide increasingly accurate solutions for increasingly singular problems as $\epsilon \to 0$. Spectral methods with ABF, which contain the dependence on the small parameter through the gauge functions, allow one to take advantage of this behavior. Incorporating the model-driven ABF into spectral numerical methods hold the potential to dramatically reduce the number of basis functions required for an accurate solution, particularly for singular problems. Therefore, incorporating ABF directly into spectral methods could significantly extend their usefulness to problems with sharp gradients and singularities, thereby addressing their primary weakness.

It is also possible that hybrid spectral methods could be developed that combine ABF and traditional basis functions within the Galerkin projection framework. The ABF could account for any singular behavior, while the traditional basis functions resolve the remaining smooth details of the solution. Similarly, a hybrid method could be developed in which ABF are used in the coordinate direction(s) containing singular behavior, and traditional spectral modes are used in the other direction(s), where the solution is smooth.

7.3 From a Reduced-Order Modeling Perspective

Instead of seeking a numerical solution of a differential equation, as with spectral numerical methods, the objective of ROM is to obtain a simplified mathematical model of a system that contains its essential features but involves a finite, and small, number of degrees-of-freedom. The modes are calculated using an optimization procedure, such as POD and its variants, applied to experimental or

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numerical data from the full system. The resulting ROM can then be used in optimization, closed-loop control, system identification, multi-disciplinary design optimization, and multiscale modeling, for example.

Interestingly, the projection methods used to obtain the ROM utilize the governing equation(s) of the system, while the POD approach to determining the basis functions does not. That is, POD is a *data-driven* method. While this is advantageous for data sets obtained from systems for which a mathematical model is not available, in a Galerkin projection context, where a mathematical model is known, this model does not come into play in formation of the POD basis functions themselves. Consequently, the primary advantage of POD analysis in generating basis functions directly from data is also its primary shortcoming in ROM.

In addition, because the optimal basis functions are determined from the actual data set, they provide the best representation of the original data with the fewest POD modes for the values of the parameter(s) used to obtain the data. However, this means that they are problem – and parameter – dependent, requiring one to obtain a new set of basis functions each time the data set changes, whether from consideration of a different dynamical model or a different set of data from the same model. RPM with ABF addresses this weakness of ROM with POD by obtaining the basis functions directly from the mathematical model of the system. Recall that the ABF incorporate both the gauge functions as well as the asymptotic solutions for each order. Consequently, both the dependence on the small parameter ϵ and the spatial coordinate x are accounted for by the ABF, which is not the case in POD analysis. Thus, this model-driven approach provides a complement and enhancement to the data-driven methods that are gaining traction in many fields today.

Finally, the majority of ROM techniques are based on linear theory, which renders them straightforward to apply but not always ideal for nonlinear system. RPM, on the other hand, accounts for the inherent nonlinearity of the system when present. Although some of the mathematical advantages of using linear system theory is lost, a dramatic reduction in the number of basis functions is anticipated when using ABF.

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