Synchronization under control in complex networks for a panic model[*]

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Abstract. After a sudden catastrophic event occurring in a population of individuals, panic can spread, persist and become more problematic than the catastrophe itself. In this paper, we explore through a computational approach the possibility to control the panic level in complex networks built with a recent behavioral model. After stating a rigorous theoretical framework, we propose a numerical investigation in order to establish the effect of the topology of the network on this control process, with randomly generated networks, and we compare the panic level for two distinct topology sets on a given network.

Keywords: Optimal control · Numerical computation · Dynamical system · Complex network · Synchronization · Panic.

1 Introduction

The aim of this paper is to explore the possibility to control the panic spreading within a population of individuals facing a catastrophic event, using a recent mathematical modeling called the Panic-Control-Reflex system (PCR system) [5], [9]. This modeling is given by a set of ordinary differential equations and reproduces the behavioral process from reflex to control behavior, with the eventuality to transit through panic, and possibly to exhibit a persistence of panic. The geographical background of the areas impacted by catastrophic events naturally leads us to consider complex networks of PCR systems [4], [6], that is, geographical networks whose nodes are coupled with multiple instances of a PCR system, with connections between those nodes, corresponding to physical displacements. In [4], it is proved that the evacuation of high risk zones towards refuge zones is a necessary and sufficient condition for the whole population in the network to return to a daily behavior, and to avoid a persistence of panic.

But this necessary evacuation can be awkward in some particular places, or even impossible. Thus it is natural to ask if it is possible to reach a synchronization state in the network, which would guaranty that panic vanishes. This question is of great interest, since it is related to the more general problem of controlling and decision making in complex systems [1], [3]. To this end, we consider the superposition of two controls, exerted on each node and on each edge in the network. Here, we propose to investigate with a computational approach the effect of the topology on the control process. In the first part, we show how to construct a complex network of PCR systems, and illustrate by a computation of randomly generated networks the risk of panic persistence. In the second part, we set the general control problem together with its performance criterion, and finally, we present a comparison of two computations obtained for two distinct topology sets of a given network.

2   Non identical PCR networks

In this section, we briefly present the Panic-Control-Reflex system (PCR), show how to construct a complex network of PCR systems, and illustrate by a computation of 800 randomly generated networks the risk of panic spreading.

2.1   Panic-Control-Reflex system

The Panic-Control-Reflex system (PCR system) is a mathematical model for human behaviors during catastrophic events, developed with the collaboration of geographers in order to better understand, predict and control the behavioral reactions of individuals facing a brutal disaster [5], [9]. It is given by the following system of ordinary differential equations

$$\begin{cases} \dot{r} = \gamma(t)q(r_m - r) - Br + f(r,\,c)rc + g(r,\,p)rp \\ \dot{c} = B_1 r - C_2 c + C_1 p - f(r,\,c)rc + h(c,\,p)cp - \varphi(t)c(b_m - b) \\ \dot{p} = B_2 r + C_2 c - C_1 p - g(r,\,p)rp - h(c,\,p)cp \\ \dot{q} = -\gamma(t)q(r_m - r) \\ \dot{b} = +\varphi(t)c(b_m - b), \end{cases} \tag{1}$$

where the unknowns r, c, p, q, b are real-valued functions defined on $\mathbb{R}$, which model the densities of individuals in *reflex*, *control*, *panic*, *daily* and *back to daily* behaviors respectively. The parameters $B_1 > 0$, $B_2 > 0$, $B = B_1 + B_2$, $C_1 \geq 0$, $C_2 \geq 0$, $r_m > 0$, and $b_m > 0$ are real coefficients, γ, φ are smooth functions of t with positive values, f, g, h smooth functions defined on $\mathbb{R}^2$ with values in $\mathbb{R}$.

When the catastrophic event occurs, individuals are brought to the reflex behavior; this evolution is modeled by the non-linear term $\gamma(t)q(r_m - r)$, in which $\gamma(t)$ corresponds to the impact of the catastrophe. Next, individuals are subject to a behavioral evolution towards control behavior or panic; this evolution is modeled by the linear terms $B_1 r$ and $B_2 r$. Additionally, contagion phenomena can act in parallel between the 3 main behavioral subgroups (reflex, control

behavior, panic); those contagion phenomena are modeled by the non-linear terms $f(r, c)rc$, $g(r, p)rp$ and $h(c, p)cp$. We emphasize that the functions f, g and h have been designed to change their signs according to the values of the proportions $\frac{r}{c}$, $\frac{r}{p}$ and $\frac{c}{p}$ [5], [9]. For instance, if the density of individuals in panic is widely greater than the density of individuals in control behavior, then $h(c, p) < 0$, which means that the contagion brings individuals in control behavior to imitate individuals in panic. In the mean time, evolution between panic and control behavior is modeled by the linear terms $C_1 p$, $C_2 c$. Finally, the return to daily behavior operates from the control behavior; it is modeled by the non-linear term $\varphi(t)c(b_m - b)$.

Remark 1. The PCR system has been considered for simulations of concrete scenarios of catastrophic events, thanks to a narrow collaboration with geographers and psychologists. The example of an earthquake in Japan is studied in [9], and the risk of tsunami on the Mediterranean coast is presented in [6]. Catastrophic events of industrial origin are also of great interest.

The parameter r_m models the maximum capacity of individuals which can be in reflex behavior. Without loss of generality, we will set $r_m = 1$ in the rest of the paper. The parameter b_m models the maximum capacity of individuals which can return to the daily behavior. We assume that this maximum capacity coincides with the total population $\Lambda = r + c + p + q + b$ involved in the catastrophic event, thus we can reduce system (1) to a 4 equations system

$$\dot{x} = \psi(t, x), \tag{2}$$

where $x = (r, c, p, q)^T$ and

$$\psi(t, x) = \begin{pmatrix} \gamma(t)q(1 - r) - Br + f(r,c)rc + g(r,p)rp \\ B_1 r - C_2 c + C_1 p - f(r,c)rc + h(c,p)cp - \varphi(t)c(r + c + p + q) \\ B_2 r + C_2 c - C_1 p - g(r,p)rp - h(c,p)cp \\ -\gamma(t)q(1 - r) \end{pmatrix}.$$

The following Theorem summaries its dynamics, and highlights the decisive role of the parameter C_1 which models the evolution from panic to control behavior. The proof is detailed in [5].

Theorem 1. *For any initial condition $x_0 \in (\mathbb{R}^+)^4$, the initial value problem*

$$\begin{cases} \dot{x} = \psi(t, x), & t > 0, \\ x(0) = x_0, \end{cases}$$

admits a unique global solution whose components are non-negative and bounded. If $C_1 > 0$, then the trivial equilibrium $0 \in \mathbb{R}^4$ is the only equilibrium, and it is globally asymptotically stable. If $C_1 = 0$, then the solution of system (2) starting from any initial condition (r_0, c_0, p_0, q_0) such that $r_0 + c_0 + p_0 + q_0 > 0$ presents a persistence of panic, that is

$$\lim_{t \to +\infty} p(t) = \bar{p} > 0.$$

2.2 Complex networks of non-identical PCR systems

When considering the geographical relief of the zone impacted by the catastrophic event, it is natural to improve the previous modeling by a spatial modeling. One way is to construct a complex network whose nodes are coupled with multiple instances of the PCR system [4], [6]. Let us consider a simple graph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ made with a finite set $\mathscr{V} = \{1, \ldots, n\}$ of n vertices, where n is a positive integer, and a finite set $\mathscr{E} = \{e_1, \ldots, e_k\}$ of k weighted edges, with non-negative weights $\varepsilon_1, \ldots, \varepsilon_k$. For each integer $l \in \{1, \ldots, k\}$, there exists a unique pair of vertices (i, j) such that e_l connects vertex i towards vertex j. We set $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_k) \in (\mathbb{R}^+)^k$, and introduce the matrix of connectivity $L(\varepsilon)$ of order n, whose off-diagonal coefficients are given by

$$L_{ji}(\varepsilon) = \begin{cases} \varepsilon_l \text{ if } e_l = (i, j) \in \mathscr{E}, \\ 0 \text{ else,} \end{cases}$$

and whose diagonal coefficients satisfy

$$L_{ii}(\varepsilon) = -\sum_{j \neq i} L_{ji}(\varepsilon).$$

Next we couple each node in the graph with an instance of the PCR system (2). Thus we set

$$x_i = (r_i, c_i, p_i, q_i)^T, \quad X = (x_1, \ldots, x_n)^T,$$
$$H = \mathrm{diag}\{1, 1, 1, 0\}, \quad HX = (Hx_1, \ldots, Hx_n)^T. \tag{3}$$

The definition of the matrix H means that individuals in daily behavior q are not concerned with migrations in the network. We allow the different instances of system (2) to admit different values of parameters, and we will especially focus on the effect of coupling PCR systems with different values of the parameter C_1, identified previously as a bifurcation parameter.

Definition 1. *We will call* node of type (1) *a node coupled with an instance of the PCR system such that $C_1 = 0$, and* node of type (2) *a node coupled with an instance of the PCR system such that $C_1 > 0$.*

A PCR network is given by

$$\dot{X} = \Psi(t, X) + L(\varepsilon)\,HX, \tag{4}$$

where $\Psi(t, X) = \left(\psi^{(1)}(t, x_1), \ldots, \psi^{(n)}(t, x_n)\right)^T$. The above index in $\psi^{(i)}(t, x_i)$, $1 \leq i \leq n$, indicates that the values of parameter C_1 can differ from one node in the network to another. The next theorem, presented in [4], establishes a necessary and sufficient condition for the solution of the PCR network (4) to converge to the trivial equilibrium, which corresponds to a global return of all individuals to the daily behavior. It is also a condition for *synchronization* in the network, since every node exhibits the same asymptotic dynamics under the considered assumptions.

Theorem 2. *For any initial condition $X_0 \in (\mathbb{R}^+)^{4n}$, the network problem*

$$\begin{cases} \dot{X} = \Psi(t, X) + L(\varepsilon)\, HX, \quad t > 0, \\ X(0) = X_0, \end{cases}$$

admits a unique solution whose components are non negative. The trivial equilibrium $0 \in \mathbb{R}^{4n}$ is the only equilibrium if and only if every node of type (1) is connected to at least one node of type (2) by an oriented chain. It that case, the trivial equilibrium is globally asymptotically stable.

2.3   Computation of the panic level for randomly generated networks

In this section, we aim to illustrate, deepen and qualify Theorem 2 with a computational approach. Thus we have generated 800 PCR networks, built with 15 nodes of type (1), 15 nodes of type (2), and N_e randomly chosen edges ($0 \leq N_e \leq 150$). This random generation can yield a great variety of topology disposals (see figure 1). The number of isolated nodes of type (1) is computed by running a path-finding algorithm implemented in the `networkx` library [2] of the `python` language. Meanwhile, the panic level $\bar{P}$ in the whole network is computed by integrating the system of ordinary differential equations (4) on a finite time interval $[0, 60]$, using a Runge-Kutta scheme of order 4 (each PCR network corresponds to a system of 120 equations).
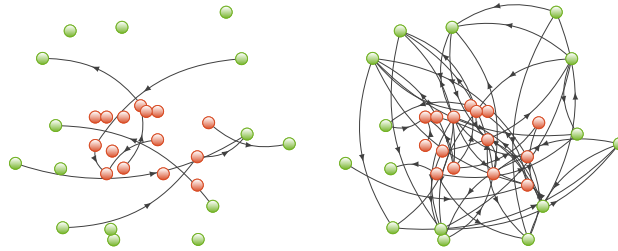

Fig. 1. Randomly generated PCR networks, built with 15 nodes of type (1) (depicted in red), 15 nodes of type (2) (depicted in green), and N_e randomly chosen edges (with $0 \leq N_e \leq 150$). Left: weakly dense topology. Right: highly dense topology.

The numerical computation has been performed on the server of the Laboratory of Applied Mathematics at University of Le Havre Normandie, in a GNU/Linux environment. The values of the parameters are $B_1 = B_2 = 0.5$,

$C_1 = 0.3$, $C_2 = 0.2$, and the imitation functions f, g, h are given by

$$f(r, c) = -\alpha_1 \xi \left(\frac{r}{c + \nu_0} \right) + \alpha_2 \xi \left(\frac{c}{r + \nu_0} \right),$$

$$g(r, p) = -\delta_1 \xi \left(\frac{r}{p + \nu_0} \right) + \delta_2 \xi \left(\frac{p}{r + \nu_0} \right),$$

$$h(c, p) = -\mu_1 \xi \left(\frac{c}{p + \nu_0} \right) + \mu_2 \xi \left(\frac{p}{c + \nu_0} \right),$$

with $\alpha_i = \delta_i = \mu_i = 0.1$, $i \in \{1, 2\}$, $\nu_0 = 10^{-2}$ and

$$\xi(s) = \begin{cases} 1 & \text{if } s < 0, \\ 0 & \text{if } s > 1, \\ \frac{1}{2} + \frac{1}{2} \cos(\pi s) & \text{else.} \end{cases}$$
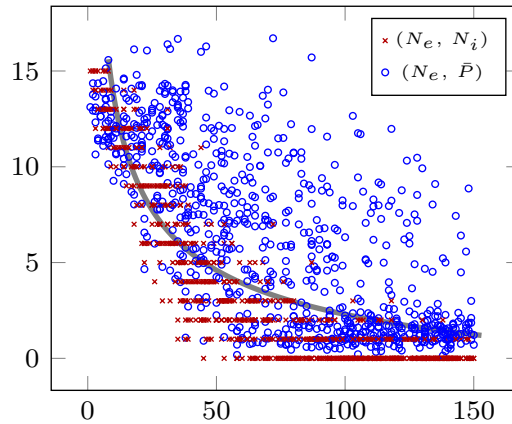

Fig. 2. Numerical results for the computation of 800 randomly generated PCR networks. For each randomly generated network, N_e denotes the number of edges, N_i the number of isolated nodes of type (1), and $\bar{P}$ the panic level in the network after a finite time. The red crosses have coordinates (N_e, N_i), and the blue circles have coordinates $(N_e, \bar{P})$. The gray line corresponds to the approximation of the cloud of blue circles by an heuristic inverse power law of the form $\bar{P} = \frac{k_1}{N_e^\nu} - k_2$.

The results are depicted in figure 2. For each randomly generated network, N_e denotes the number of edges, N_i the number of isolated nodes of type (1), and $\bar{P}$ the panic level in the network after a finite time T, defined by

$$\bar{P} = \sum_{i=1}^{n} p_i(T).$$

The red crosses have coordinates (N_e, N_i), and the blue circles have coordinates $(N_e, \bar{P})$. Obviously, 15 edges could be sufficient to evacuate 15 nodes of type (1) towards nodes of type (2). However, we remark that the number of isolated nodes of type (1) can be relatively high, even with a dense topology. For example, some of the generated networks admit 70 edges and 5 isolated nodes of type (1). Furthermore, the panic level can remain high after a large time, even if the number of evacuated nodes of type (1) is law, which means that the asymptotic result of Theorem 2 has to be qualified. In other words, a dense topology is not an absolute warranty for the network to converge to the trivial equilibrium that corresponds to the return of all individuals to a daily behavior. Finally, the cloud of blue circles can be approximated by an heuristic inverse power law (see gray curve in figure 2) of the form

$$\bar{P} = \frac{k_1}{N_e^{\nu}} - k_2,$$

with positive coefficients k_1, k_2, ν, which can be used for prediction. This numerical computation motivates the introduction of an optimal control process in order to limit the panic level at a reasonable level.

3   Optimal control problem

In this section, we introduce an optimal control problem, related to the expected synchronization state of PCR networks, and show the existence of a solution to that problem.

3.1   Synchronization under control

In order to avoid the possible persistence of panic pointed above, we consider a multiple control $u = (u_0, u_1, \ldots, u_k)$ for the network problem (4), in which u_0 models an internal control introduced on each node in order to facilitate the evolution from p to c (see figure 3), and $(u_1, \ldots, u_k)$ corresponds to an external control exerted in order to increase the coupling strength along each edge in the network. Thus we consider the following general control problem

$$\dot{X} = \tilde{\Psi}(t, X, u) + \tilde{L}(\varepsilon, u) \, HX, \tag{5}$$

where $\tilde{\Psi}(t, X, u) = \left(\tilde{\psi}(t, x_1, u), \ldots, \tilde{\psi}(t, x_n, u)\right)^T$, with

$$\tilde{\psi}(t, x, u) = \begin{pmatrix} -Br + \gamma(t)q(1-r) + f(r,c)rc + g(r,p)rp \\ B_1 r - C_2 c + (C_1 + u_0)p - f(r,c)rc + h(c,p)cp - \varphi(t)c\theta \\ B_2 r + C_2 c - (C_1 + u_0)p - g(r,p)rp - h(c,p)cp \\ -\gamma(t)q(1-r) \end{pmatrix},$$

where $\theta = r + c + p + q$ and the matrix $\tilde{L}(\varepsilon, u)$ is defined by

$$\tilde{L}(\varepsilon, u) = L(\varepsilon_1 + u_1, \ldots, \varepsilon_k + u_k). \tag{6}$$

The internal control u_0 appears in the system through the term $(C_1 + u_0)p$, whereas the external controls $(u_1, \ldots, u_k)$ appear through the coupling terms $(\varepsilon_i + u_i)r_i$, $(\varepsilon_i + u_i)c_i$, $(\varepsilon_i + u_i)p_i$, $1 \le i \le k$.

Remark 2. For the internal control u_0, awareness campaigns can be organized in the areas for which a high potential of catastrophic event is identified; those campaigns can be integrated to the educational programs to better prepare individuals to the known risks. External controls $u_1, \ldots, u_k$ can be made through rescue services, in order to clear some hindered avenue, or to repair any urban installation. We emphasize that it is a work in progress, in collaboration with geographers and psychologists, to establish an exhaustive list of possible actions in concordance with the mathematical control functions $(u_0, u_1, \ldots, u_k)$.

The aim of introducing the control $u = (u_0, u_1, \ldots, u_k)$ is to limit panic at a reasonable level in complex networks for which we can predict, using Theorem 2, a persistence of panic on nodes of type (1). However, we will see below that the solution of the control problem satisfies $q_i(t) > 0$ for all $t > 0$, $1 \le i \le n$, and for each initial condition such that $q_i(0) > 0$, $1 \le i \le n$ (see Equation (10)). This demonstrates that the trivial equilibrium $0 \in \mathbb{R}^{4n}$ cannot be reached in a finite time, when starting from such initial conditions. A more pragmatic goal would be to reach a neighborhood $\mathscr{N}$ of the trivial equilibrium. For instance, we can look for a multiple control so that the panic level is limited under a given proportion of the total population after a finite time.

3.2   Performance criterion

In what follows, we denote by $\mathscr{U}$ the set of admissible control functions, composed with Lebesgue-integrable functions $u = (u_0, u_1, \ldots, u_k)$ for which there exists $T > 0$ such that u is defined on $[0, T]$ with values in $K = [0, 1]^{k+1}$. Note that T may depend on u. Let $u = (u_0, u_1, \ldots, u_k) \in \mathscr{U}$ denote an admissible control for the general control problem (5). Applying this multiple control, we aim to reach a neighborhood $\mathscr{N}$ of the trivial equilibrium $0 \in \mathbb{R}^{4n}$ which corresponds to a synchronization state of the network. Additionally, we would like to minimize on $\mathscr{U}$ the performance index:

$$J(X_0, u, T) = \int_0^T \left[\sum_{i=1}^n p_i^2(t) + u_0^2(t) + \sum_{l=1}^k u_l^2(t) \right] \mathrm{d}t, \tag{7}$$

which models the wish to limit the level of panic during the control process, while mobilizing the less rescue services to operate during the catastrophic event.

Finally, we can state the optimal control problem for the complex network of non-identical PCR systems. The problem is to find a pair (X, u) defined on some interval $[0, T]$, such that $u \in \mathscr{U}$ and

$$\begin{cases} \dot{X} = \tilde{\Psi}(t, X, u) + \tilde{L}(\varepsilon, u) \, HX, \quad t > 0, \\ X(0) = X_0, \quad X(T) \in \mathscr{N}, \\ \min_{u \in \mathscr{U}} J(X_0, u, T), \end{cases} \tag{8}$$

where X_0 is a given initial datum in $(\mathbb{R}^+)^{4n}$, and $\mathscr{N}$ denotes a neighborhood of the trivial equilibrium $0 \in \mathbb{R}^{4n}$.

3.3 Existence of an optimal control

Theorem 3. *Let $u = (u_0, u_1, \ldots, u_k)$ denote a multiple control with non-negative values. We assume that u is continuous in t and bounded. Then for any $X_0 \in (\mathbb{R}^+)^{4n}$, the initial value problem*

$$\begin{cases} \dot{X} = \tilde{\Psi}(t, X, u) + \tilde{L}(\varepsilon, u)\, HX, \quad t > 0, \\ X(0) = X_0, \end{cases} \tag{9}$$

admits a unique global solution whose components are non-negative and bounded. Furthermore, the optimal control problem (8) for the complex network of PCR systems admits a solution (X_0, u^*), with $u^* \in \mathscr{U}$ minimizing (7).

Proof. Given any initial condition $X_0 \in (\mathbb{R}^+)^{4n}$, existence and uniqueness of a local in time solution $X(t, X_0)$ to problem (9), defined on some interval $[0, \tau]$ with $\tau > 0$, are a straightforward consequence of Cauchy-Lipschitz Theorem [8].

Non-negativity. In order to prove the non-negativity property, we introduce a modified problem as follows. For $\hat{x} = (\hat{r}, \hat{c}, \hat{p}, \hat{q}) \in \mathbb{R}^4$, define

$$\hat{\psi}(\hat{x}, u) = \left(\hat{\psi}_1(\hat{x}, u), \hat{\psi}_2(\hat{x}, u), \hat{\psi}_3(\hat{x}, u), \hat{\psi}_4(\hat{x}, u) \right)^T \quad \text{with}$$

$$\hat{\psi}_1(\hat{x}, u) = +\gamma\hat{q}(1 - \hat{r}) - B\hat{r} + f(\hat{r}, \hat{c})\hat{r}\hat{c} + g(\hat{r}, \hat{p})\hat{r}\hat{p}$$

$$\hat{\psi}_2(\hat{x}, u) = B_1\hat{r} - C_2\hat{c} + (C_1 + u_0)\,|\hat{p}| - f(\hat{r}, \hat{c})\hat{r}\hat{c} + h(\hat{c}, \hat{p})\hat{c}\hat{p} - \varphi\hat{c}\hat{\theta}$$

$$\hat{\psi}_3(\hat{x}, u) = B_2\hat{r} + C_2\hat{c} - (C_1 + u_0)\hat{p} - g(\hat{r}, \hat{p})\hat{r}\hat{p} - h(\hat{c}, \hat{p})\hat{c}\hat{p}$$

$$\hat{\psi}_4(\hat{x}, u) = -\gamma\hat{q}(1 - \hat{r}),$$

where $\hat{\theta} = \hat{r} + \hat{c} + \hat{p} + \hat{q}$ (we omit the dependence in t in order to lighten our notations). Next, for $\hat{X} = (\hat{x}_1, \ldots, \hat{x}_n)$, consider the modified network problem

$$\dot{\hat{x}}_i = \hat{\psi}(\hat{x}_i, u) + \sum_{\substack{k=1 \\ k \neq i}}^{n} \tilde{L}_{ik} H\, |\hat{x}_k| - \sum_{\substack{k=1 \\ k \neq i}}^{n} \tilde{L}_{ik} H\hat{x}_i, \quad 1 \leq i \leq n,$$

with the notation $|\hat{x}| = (|\hat{r}|, |\hat{c}|, |\hat{p}|, |\hat{q}|)$, and $\tilde{L}_{ik}$ denoting the coefficient of index (i, k) in the matrix $\tilde{L} = \tilde{L}(\varepsilon, u)$. For the same initial condition X_0 as in the non-modified problem, existence and uniqueness of a local in time solution $\hat{X}(t, X_0)$ defined on some interval $[0, \hat{\tau}]$ with $\hat{\tau} > 0$, are also obtained by Cauchy-Lipschitz Theorem.

Now, we recall that $H = \text{diag}\,\{1, 1, 1, 0\}$ (see equation (3)), which means that the $\hat{q}_i$ components, $1 \leq i \leq n$, are not coupled. It follows that

$$\hat{q}_i(t) = \hat{q}_i(0)e^{-\int_0^t \gamma(s)(1 - \hat{r}_i(s))\mathrm{d}s}, \quad t \in [0, \hat{\tau}], \quad 1 \leq i \leq n, \tag{10}$$

which implies $\hat{q}_i(t) \geq 0$ for all $t \in [0, \hat{\tau}]$ and $1 \leq i \leq n$, since $\hat{q}_i(0) \geq 0$. Let us next examine the non-negativity of other components $\hat{r}_i$, $\hat{c}_i$ and $\hat{p}_i$, $1 \leq i \leq n$. We employ a truncation method presented in [12]. Define the real-valued function χ on $\mathbb{R}$ by

$$\chi(s) = \begin{cases} 0 & \text{if } s > 0, \\ \frac{1}{2}s^2 & \text{if } s \leq 0. \end{cases}$$

The function χ is of class $\mathscr{C}^1$ on $\mathbb{R}$, with $\chi'(s) = 0$ if $s > 0$, $\chi'(s) = s$ if $s \leq 0$. Furthermore, it enjoys the properties

$$\chi(s) \geq 0, \quad \chi'(s) \leq 0, \quad 0 \leq \chi'(s)\, s = 2\chi(s), \quad \forall s \in \mathbb{R}. \tag{11}$$

Next we introduce for each i such that $1 \leq i \leq n$ the function ρ_i defined by

$$\rho_i(t) = \chi\big(\hat{r}_i(t)\big), \quad t \in [0, \hat{\tau}].$$

We easily prove that $\rho_i(t) = 0$ for all $t \in [0, \hat{\tau}]$, thus $\hat{r}_i(t) \geq 0$ for all $t \in [0, \hat{\tau}]$. Applying the same method leads to $\hat{c}_i(t) \geq 0$ and $\hat{p}_i(t) \geq 0$ for all $t \in [0, \hat{\tau}]$.

Hence, the components of the solution $\hat{X}(t, X_0)$ of the modified problem are non-negative, so $\hat{X}(t, X_0)$ is also a solution of the initial non-modified problem on $[0, \hat{\tau}]$. By uniqueness, we have $\hat{X}(t, X_0) = X(t, X_0)$ on $[0, \tau] \cap [0, \hat{\tau}]$. Finally, it is seen that $\tau = \hat{\tau}$, thus we have proved the non-negativity of the components of $X(t, X_0)$ on $[0, \tau]$.

Boundedness. In order to prove the boundedness of the solution, we introduce the function θ defined by

$$\theta(t) = \sum_{i=1}^{n} \big[r_i(t) + c_i(t) + p_i(t) + q_i(t)\big], \quad t \in [0, \tau].$$

After basic computations, we prove that $\dot{\theta}(t) \leq 0$ for all $t \in [0, \tau]$. This implies that $\theta(t) \leq \theta(0)$ for all $t > 0$, which guarantees the boundedness of the solution.

Optimal control problem. The existence of an optimal control follows from Theorem III.4.1 in [7], since $\tilde{\Psi}$ and $\tilde{L}$ are linear in u, K is convex, and J is defined by integrating a convex function. □

4   Numerical computation of an optimal control

We end our paper with two numerical computations of optimal controls for two distinct topology disposals of a given PCR network of 8 nodes, composed with 4 nodes of type (1) and 4 nodes of type (2). For each network, the values of the parameters are $B_1 = B_2 = 0.3$, $C_1 = 0.1$, $C_2 = 0.2$, $\alpha_i = \delta_i = \mu_i = 0.01$, and the neighborhood $\mathscr{N}$ of the trivial equilibrium which is aimed to be reached after a finite time is defined by $\mathscr{N} = [0, 10^{-1}]^{32}$. The values of other parameters are unchanged. This second computation has also been performed on the server of the Laboratory of Applied Mathematics at University of Le Havre Normandie, using the free and open-source optimal control software `BOCOP` developed at the INRIA [11].

Remark 3. The two topology disposals presented in this section are inspired by the shape of geographical networks, used for the simulation of panic spreading in the case of a tsunami of low intensity on the Mediterranean coast [6]: nodes of type (1) correspond to exposed beaches, while nodes of type (2) model refuge zones in the city center.
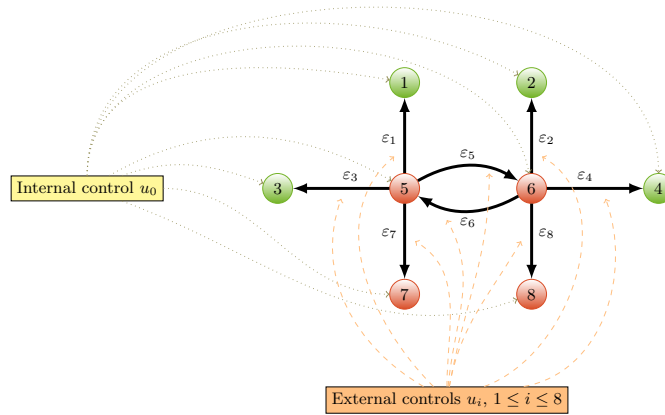
4.1   Symmetric topology

Fig. 3. Symmetric topology for an 8 nodes PCR network, composed with 4 nodes of type (1) (depicted in red) and 4 nodes of type (2) (depicted in green). Nodes 7 and 8 are not evacuated towards any node of type (2).

First, we examine a symmetric topology shown in figure 3. Nodes 7 and 8 are not evacuated towards any node of type (2), thus we can predict, by virtue of Theorem 2, a persistence of panic on this network. We apply a multiple control $(u_0, u_1, \ldots, u_8)$ on this network, and compute an optimal control with respect to the performance criterion J given by (7). The results of the computation are presented in figure 4.

In this first case, the value of the performance criterion is $J_{min} \simeq 4.1734$, and the final time is $T_f \simeq 14.11$. We observe that the main control that has to be exerted is the internal control u_0. This internal control favors the behavioral evolution from panic p to control behavior c on each node. Meanwhile, the controls u_i, $1 \leq i \leq 4$, exerted along the edges e_i, $1 \leq i \leq 4$, facilitate the evacuation of individuals of nodes 5 and 6. Finally, the controls u_i, $i \in \{7, 8\}$, exerted along the edges e_i, $i \in \{7, 8\}$, are of very low intensity. Roughly speaking, it seems unnecessary to displace individuals from a node of type (1) towards another node of type (1), if the latter is not itself evacuated towards any node of type (2).
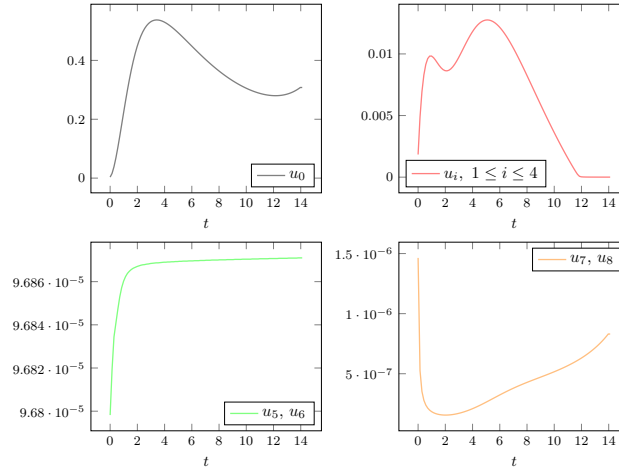
Fig. 4. Numerical results for the computation of an optimal control for the 8 nodes PCR network shown in figure 3.

4.2 Asymmetric topology

Next, we consider an asymmetric topology presented in figure 5, motivated by the well-known symmetry-breaking effect [10]. This topology is of great interest, since it exhibits the cohabitation of a cycle, given by the path $2-5-6-8-4-2$, and a tree, centered at node 5, with leaves 1, 3, 6, 7. With this topology, node 7 is the only non-evacuated node of type (1).
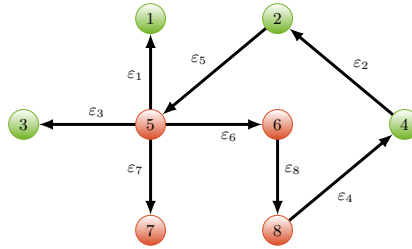

Fig. 5. Asymmetric topology for an 8 nodes PCR network, composed with 4 nodes of type (1) (depicted in red) and 4 nodes of type (2) (depicted in green). Node 7 is not evacuated towards any node of type (2).

The numerical results are shown in figure 6. In this second case, the value of the performance criterion is $J_{min} \simeq 4.2187$, and the final time is $T_f \simeq 15.17$. Once again, we observe that the main control that has to be exerted is the internal control u_0. The controls u_1 and u_3 are equal to each other. Surprisingly, the control u_6 exerted between the nodes 5 and 6 which are both of type (1) is of
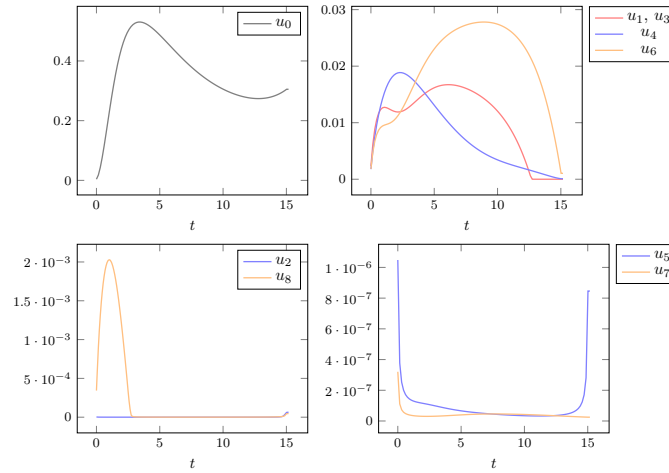
Fig. 6. Numerical results for the computation of an optimal control for the 8 nodes PCR network shown in figure 5.

greater intensity than in the first case, which seems to be a consequence of the effect of the cycle $2 - 5 - 6 - 8 - 4 - 2$. Finally, the control u_5 exerted along the edge e_5 admits very low values, since it can worsen the panic level to displace individuals from the node 2 of type (2) towards the node 5 of type (1); this low value of the control u_5 seems to "cut" the cycle $2 - 5 - 6 - 8 - 4 - 2$ between nodes 2 and 5.

Conclusion

In this paper, we have presented, through a computational approach, the possibility to reach a synchronization state in complex networks of PCR systems, for which we can predict a persistence of panic, by applying an optimal control process. Numerical computations qualify theoretical results, and highlight the decisive role of the internal control exerted on each node of the network. In a future work, we aim to deepen our collaboration with geographers and psychologists, in order to extend our study to reaction-diffusion networks of PCR systems, by taking into account the effect of a local diffusion of panic by random walk.

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