## Mathematical Modelling of Wormhole-routed x-Folded TM Topology in the Presence of Uniform Traffic

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**Abstract.** Recently, x-Folded TM topology was introduced as a desirable design in $k$-ary $n$-cube networks due to the low diameter and short average distance. In this article, we propose a mathematical model to predict the average network delay for $(k \times k)$ x-Folded TM in the presence of uniform traffic pattern. Our model accurately formulates the applied traffic pattern over network virtual channels based on the average distance and number of nodes. The mathematical results indicate that the average network delay for x-Folded TM topology is reduced when compared with other topologies in the presence of uniform traffic pattern. Finally, the results obtained from simulation experiments confirm that the mathematical model exhibits a significant degree of accuracy for x-Folded TM topology under the traffic pattern even in varied virtual channels.

**Keywords:** Interconnection networks, x-Folded TM topology, virtual channel, mathematical model, delay.

# 1 Introduction

One of the critical components in a multicomputer is the interconnection network, which significantly influences multicomputer performance. The three significant factors for these network performance are introduced as topology, routing algorithm and switching method. These factors are effectively used to determine network performance. The designs of nodes and the connection to channels are described in the network topology. To select a path between the source and destination, a routing algorithm is employed to select network messages in order to specify the path across the network. Using the switching method, the allocation of channels and buffer resources for a message across the network is established [1].

A number of studies have disclosed the fact that the advantages of designing variable topologies fall under different traffic patterns. The significant advantage of the mathematical model over simulation is that it can be used to obtain performance results for large systems and behaviour under network configurations and working conditions, which may not be feasi-

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ble for study using simulation due to the disproportionate computation demands. There are several mathematical models for $k$-ary $n$-cubes topologies ($n$ is the dimension and $k$ is the number of nodes in each dimension) under different traffic patterns based on the literature review [3]. In light of this review, this article proposes the mathematical model of the x-Folded TM topology. x-Folded TM [7] is introduced as a low diameter network which can outperform a similar network size in terms of throughput and delay. Using mathematical models in this article, we can see the effect of different topologies and virtual channels on the network performance including the x-Folded TM topology properties and applied traffic patterns.Since a uniform traffic pattern has been used in previous studies and recently non-uniform traffic has been presented [4,5,8,9], unifrom traffic pattern has been employed for evaluation in this study.

## 2 x-Folded TM Topology

x-Folded TM is a new wormhole-routed topology created by folding the TM topology [10,11] based on the imaginary $x$-axis when $k \geq 3$ and $n = 2$. The definition of x-Folded TM topology is as follows:

**Definition 1.** In x-Folded TM, node $(x, y)$ is a valid node if $0 \leq x \leq (k-1)$ and $0 \leq y \leq (k-1)$. Along $x$-axis, the nodes connecting to node $(x, y)$ are: $(x+1, y)$ if $x < (k-1)$ and $(x-1, y)$ if $x > 0$. Along the $y$-axis, nodes $(x, y+1)$ if $y < (k-1)$ and $(x, y-1)$ if $y > 0$ are connected to node $(x, y)$. Then, node $(x, y)$ is removed from the x-Folded TM if $(x + y) \mod k = 0$ or $1$ or ... or $(k-3)$, where $x > y$ and $(k-3) \leq x \leq (k-1)$ and $1 \leq y \leq (k-2)$. In addition, there is no link between two nodes $(x, y)$ and $(x+1, y)$ if $x = i$ and $y = i+1$, when $k$ is even and $i = \frac{k}{2} - 1$.

x-Folded TM topology has a node structure similar to TM topology, except a number of the nodes are shared from top to bottom. Figures 1 (a) and (b) show x-Folded TM topology for an $(k \times k)$ interconnection network where $k$ is even and odd respectively. The purple nodes in this figure are representative of the shared nodes between red and blue nodes. Several researchers have proposed mathematical models under different permutations, which have been reported in previous studies. The model developed here uses the main advantages of the proposed model in [8] to derive the mathematical model for the x-Folded TM topology in the presence of uniform and non-uniform traffic patterns.

## 3 Mathematical Model

This section describes the derivation of the mathematical model based on the introduced assumptions in Table 1.

The model has restricted the attention to x-Folded TM topology, where $n = 2$ is the network dimension and $k \geq 3$ is radix or the number of nodes

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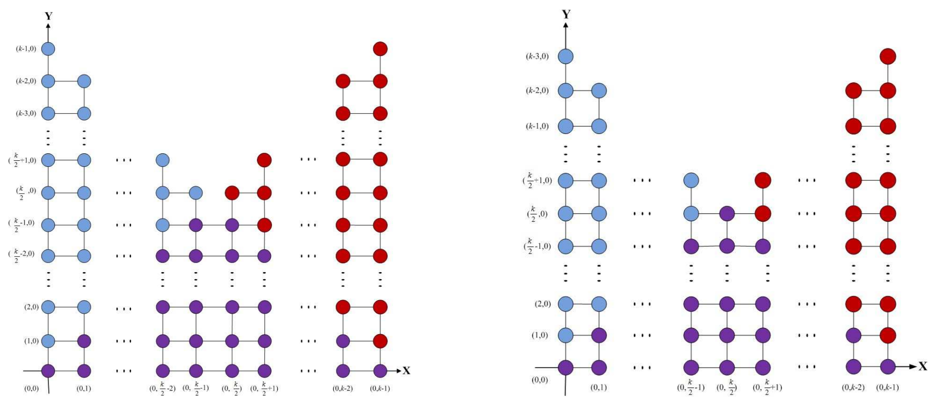


(a) $k$ is even (b) $k$ is odd

Fig. 1: $(k \times k)$ x-Folded TM Topology

Table 1: Mathematical Assumptions.

| Parameter | Description |
|---|---|
| **Mean Rate** ($\lambda$) | The traffic is generated across the network nodes independently and follow a Poisson process with a mean rate. |
| **Network size** ($N$) | $N = 8 \times 8$ ($k$-ary $n$-cube where $k$=8 and $n$=2) |
| **Message length** ($L$) | The length of a message is $L$ flits (A packet is broken into small pieces called flits which are flow control digits) and each flit is transmitted from source to destination in one cycle across the network. |
| **Virtual Channels** ($VC$) | VCs = 2 or 8 are used per physical channel. |

in each dimension. Generally the relation between dimension, radix and a number of nodes (*N*) is

$$N = k^n \qquad or \qquad (n = log_k N). \tag{1}$$

In the $k$-ary $n$-cube, the average delay is a combination of the average network delay ($\bar{S}$) and the average waiting time ($\bar{W}$) of the source node while the average degree of VCs at each physical channel also influences the average delay and is scaled by a parameter ($\bar{V}$). Thus, the average delay is

$$Average\ Delay = (\bar{S} + \bar{W})\bar{V}. \tag{2}$$

In [2], the average distance along one dimension is defined as $AD$ in that it is multiplied by a number of dimensions, $n$, for the whole network and is defined $\bar{d}$ as

$$\bar{d} = n \times AD. \tag{3}$$

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In order to demonstrate Eq. (3), $AD$ can be obtained according to Theorem 1. Although we need to present Lemma 1 and Preposition 1 before proving Theorem 1.

**Lemma 1.** *For an x-Folded TM, let I(i) be the number of the nodes that are i nodes away from the source node (0), which can be calculated by using Eq.(4),*

$$I(i) = i + 1 \qquad\qquad \forall 1 \leq i \leq (k-1). \tag{4}$$

*Proof.* For $1 \leq i \leq (k-1)$, $I(i)$ has been defined as the number of the nodes that are $i$ nodes away from node (0). The diagram for $(8 \times 8)$ x-Folded TM topology in Figure 2 used to show the distance between the assumed source node and the other nodes by dashed lines. For simplicity, all nodes are labelled from 0 to ($k^2$-1) in this figure. The node with coordinates (0, 0) has been labelled node (0) and introduced as the assumed source node. For example, $I(1) = 2$, i.e., the number of nodes that are one node away from node (0) is two nodes. In other words, this means that the node (0) has two neighbours. □

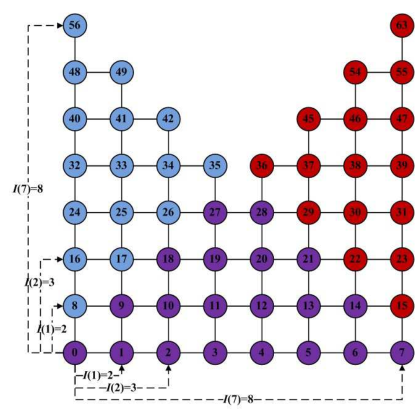


Fig. 2: Illustration of the Distance of Nodes from node (0) as source node in $(8 \times 8)$ x-Folded TM topology.

**Theorem 1.** *The average distance of x-Folded TM topology is*

$$AD_k = \begin{cases} 1 & if \quad k = 3, \\ \frac{AD_{k-1} + k(k-1)}{k^2 - 1} & if \quad k \geq 4. \end{cases} \tag{5}$$

*Proof.* Using Proposition 1, the average distance of the $k$-ary $n$-cube topologies is denoted by $AD$. Substituting Eq. (4) into average distance of an $k$-ary $n$-cube topology along one dimension in [] the average distance of x-Folded TM topology ($AD_k$) appears in Eq. (5) for different $k$. □

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Under uniform traffic pattern, messages arrive at network channels at different injection rates. The received injection rate of messages for each channel, $\lambda$, is gained by dividing the total channel rates by the total number of channels. The uniform's injection rate, $\lambda_u$, can be found using

$$\lambda_u = \frac{1}{2} \cdot (\lambda \times AD_k). \tag{6}$$

**Proposition 1.** *The number of the nodes for an x-Folded TM can be written as,*

$$N' = a_1 \times N + a_2. \tag{7}$$

It is computed according to a linear model, which has been presented in Eq. (7) where the initial values are $a_1 = 0.8005$, $a_2 = 0.9786$, and *n*=2.

**Theorem 2.** [8] *The number of channels that are j nodes away from a given node in a k-ary n-cube topology is*

$$C_j = \sum_{l=0}^{n-1} \sum_{t=0}^{n-1} (-1)^t (n-l) \binom{n}{l} \binom{n-l}{t} \binom{j-t(k-1)-1}{n-l-1}. \tag{8}$$

*Proof.* By referring to the combinatorial theory, this theorem has been proven in [8]. □

Under uniform traffic pattern, the average network delay for each channel in different dimensions is equal to message lengths at the beginning, $\bar{S}_0$. The studies in [9,8] provide the used notation to determine the quantities of the average waiting time, $W_{c_i}$ and the probability of blocking, $P_{b_i}$, at dimension $i$. Considering the average delay, the average network delay can be found using

$$\bar{S}_i = \bar{S}_{i-1} + AD_k(1 + W_{c_i} P_{b_i}) \qquad (1 \leq i \leq n), \tag{9}$$

In addition to computing $\bar{S}$ to develop the mathematical model, the waiting time ($W$) for a message under uniform traffic pattern is computed as

$$W_j = \frac{\frac{\lambda}{V} S_j^2 (1 + \frac{(S_j - S_{j-1})^2}{S_j^2})}{2(1 - \frac{\lambda}{V} S_j)}. \tag{10}$$

$\bar{W}$ is a function of the average waiting time under different possible values $(1 \leq j \leq n(k-1))$. $\bar{V}$ is the average degree of VCs under the traffic pattern. It should also be noted that $p_{v_j}$ is the probability used to determine whether the VCs are busy at the physical channel using a Markovian model [9]. The bandwidth is shared between multiple VCs in each physical channel. The average of all the possible values at a given physical channel is computed using

$$\bar{V} = \sum_{j=1}^{n(k-1)} \frac{\sum_{v=1}^{VC} v^2 p_{v_j}}{\sum_{v=1}^{VC} v p_{v_j}}. \tag{11}$$

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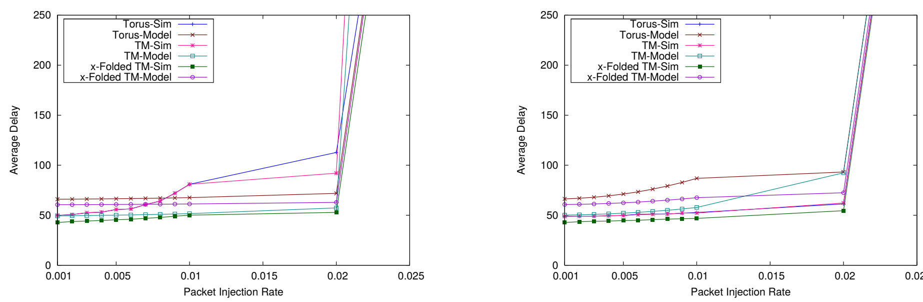
Consequently, the average network delay between source node and destination node can be written simply as Eq. (2). We presented all of the equations for the mathematical model under uniform traffic pattern to evaluate the x-Folded TM topology performance.

## 4 Performance Evaluation

The validation results in this section prove that the predictions of the mathematical model with similar assumptions are accurate in terms of the average delay (cycles) with an increase in the packet injection rate (flits/node/cycle). The presented simulation results validate the accuracy of the results obtained from the mathematical model for different topologies. The mathematical model for the x-Folded TM topology has been validated through the discrete-event simulation using the Booksim 2.0 simulator [6].




(a) *VC*=2, *L*=8 (b) *VC*=8, *L*=8

Fig. 3: Average delay in the presence of uniform traffic pattern.

The average delay curves for three topologies are illustrated in Figure 4. The used traffic pattern is uniform to generate destination nodes randomly. Figure 4 reveals the similar saturation time in different numbers of VCs and the effectiveness of x-Folded TM topology on decreasing the average delay compared with the results obtained by Torus and TM under uniform traffic pattern. Figure 4 (a) illustrates the predicted results by the mathematical model for different topologies. These have been studied for VCs=2 and the same packet size. We employ Torus and TM topologies as benchmarks and represent their mathematical models to show the superiority of x-Folded TM topology on the average delay as a new topology in interconnection networks. Figure 4 (b) illustrates the average delay in terms of the packet injection rate in VCs=8 for different topologies. Both figures reveal the less than average delay for x-Folded TM topology in different numbers of VCs until the saturation time.

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## 5 Conclusion

This article presents a mathematical model for the computation of the average network delay in wormhole-routed Torus, TM and x-Folded TM topologies in the presence of uniform and non-uniform traffic patterns. We explore the mathematical model for x-Folded TM topology to present its novelty by reducing the average delay. This model is validated and deemed accurate by simulation with less than 5% difference in the average delay. We believe that x-Folded TM topology will be applicable for the future studies in high-performance interconnection networks.


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## References


1. V. S. Adve, and M. K. Vernon, "Performance Analysis of Mesh Interconnection Networks with Deterministic Routing," *IEEE Trans. Parallel Distrib. Syst.*, vol. 5, no. 3, pp. 225-246, 1994.
2. A. Agarwal, "Limits on interconnection network performance," *IEEE Trans. Parallel Distrib. Syst.*, vol. 2, no. 4, pp. 398-412, 1991.
3. S. Gajin, and Z. Jovanovic, "An accurate performance model for network-onchip and multicomputer interconnection networks," *Journal of Parallel and Distributed Computing*, vol. 72, no. 10, pp. 1280-1294, 2012.
4. H. Sarbazi-Azad, L. M. Mackenzie, and M. Ould-Khaoua, "A performance model of adaptive routing in k-ary n-cubes with matrix-transpose traffic." in IEEE International Conference on Parallel Processing, 2000.
5. W. -J. Guan, W. Tsai, and D. Blough, "An analytical model for wormhole routing in multicomputer interconnection networks," *IEEE Comput. Soc. Press,* pp. 650-654, 1993.
6. N. Jiang, D. U. Becker, G. Michelogiannakis, J. Balfour, B. Towles, D. E. Shaw, and W. J. Dally, "A detailed and flexible cycle-accurate Network-on-Chip simulator," in 2013 IEEE International Symposium on Performance Analysis of Systems and Software (ISPASS), pp. 86-96, 2013.
7. M. Moudi, M. Othman, K. Y. Lun, A. R. Abdul Rahiman, "x-Folded TM: an efficient topology for interconnection networks," *Journal of Network and Computer Applications*, vol. 73, pp. 27-34, 2016.
8. H. Sarbazi-Azad, M. Ould-Khaoua, and L. M. Mackenzie, "Analytical Modelling of Wormhole-Routed k-Ary n-Cubes in the Presence of Hot-Spot Traffic," *IEEE Trans. Comput.*, vol. 50, no. 7, pp. 623-634, 2001.
9. H. Sarbazi-Azad, "A mathematical model of deterministic wormhole routing in hypercube multicomputers using virtual channels," *Applied Mathematical Modelling*, vol. 27, pp. 943-953, 2003.
10. X. Wang, D. Xiang, Z. Yu, "TM: a new and simple topology for interconnection networks," *The Journal of Supercomputing*, vol. 66, no. 1, pp. 514-538, 2013.
11. X. Wang, D. Xiang, Z. Yu, "A cost-effective interconnect architecture for interconnection network," *IETE Journal of Research*, vol. 59, no. 2, pp.109-117, 2013.



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