Novel Monte Carlo Algorithm for Solving Singular Linear Systems

Behrouz Fathi Vajargah[1] , Vassil Alexandrov[2],
Samaneh Javadi[3] and Ali Hadian[4]

[1] Department of Statistics, University of Guilan, P. O. Box 1914, Rasht, Iran.
fathi@guilan.ac.ir
[2] ICREA-Barcelona Supercomputing Centre. vassil.alexandrov@bsc.es
[3] Faculty of Technology and Engineering, East of Guilan, University of Guilan, P. O. Box 44891-63157, Roudsar, Iran. s.javadi62@gmail.com
[4] Department of Mathematics, University of Guilan, P. O. Box 1914, Rasht, Iran.
ahg.info2003@gmail.com

Abstract. A new Monte Carlo algorithm for solving singular linear systems of equations is introduced. In fact, we consider the convergence of resolvent operator R_λ and we construct an algorithm based on the mapping of the spectral parameter λ. The approach is applied to systems with singular matrices. For such matrices we show that fairly high accuracy can be obtained.

Keywords: Monte Carlo, Markov Chain, resolvent operator.

1 Introduction

Consider the linear system $Tx = b$ where $T \in \mathbb{R}^{n\times n}$ is a nonsingular matrix and b, T are given. If we consider $L = I - T$, then

$$x = Lx + b. \tag{1}$$

The iterative form of (1) is $x^{(k+1)} = Lx^{(k)} + b, k = 0, 1, 2....$ Let us have now $x^0 = 0, L^0 = I$, we have

$$x^{(k+1)} = \Sigma_{m=0}^{k} L^m b. \tag{2}$$

If $\|L\| < 1$, then $x^{(k)}$ tends to the unique solution x [7]. In fact, the solution of (1) can be obtained by using the iterations

$$\lim_{k\to\infty} x^{(k)} = \lim_{k\to\infty} \Sigma_{m=0}^{k} L^m b = (I - L)^{-1} b = T^{-1} b = x. \tag{3}$$

We consider the stochastic approach. Suppose that we have a Markov chain given by:

$$\alpha_0 \to \alpha_1 \to ... \to \alpha_k$$

where $\alpha_i,\ i = 0, 1, 2, 3..., k$ belongs to the state space $\{1, 2, ..., n\}$. Then $\alpha,\ \beta \in \{1, 2..., n\}$, $p_\alpha = P(\alpha_0 = \alpha)$ is the probability that the Markov chain starts at

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state α and $p_{\alpha\beta} = P(\alpha_{i+1} = \beta|\alpha_i = \alpha)$ is the transition probability from state α to β. The set of all probabilities $p_{\alpha\beta}$ defines a transition probability matrix $[p_{\alpha\beta}]$. We say that the distribution $[p_1, p_2.., p_n]^t$ is acceptable for a given vector h, and the distribution $p_{\alpha\beta}$ is acceptable for matrix L, if $p_\alpha > 0$ when $h_\alpha \neq 0$, and $p_\alpha \geq 0$ when $h_\alpha = 0$, and $p_{\alpha\beta} > 0$ when $l_{\alpha\beta} \neq 0$ and $p_{\alpha\beta} \geq 0$ when $l_{\alpha\beta} = 0$ respectively. We assume

$$\Sigma_{\alpha=1}^{n} p_\alpha = 1, \quad \Sigma_{\beta=1}^{n} p_{\alpha\beta} = 1$$

for all $\alpha = 1, 2.., n$. The random variable whose mathematical expectation is equal to $\langle x, h\rangle$ is given by the following expression

$$\theta(h) = \frac{h_{\alpha_0}}{p_{\alpha_0}} \Sigma_{j=0}^{\infty} W_j b_{\alpha_j} \tag{4}$$

where $W_0 = 1, W_j = W_{j-1} \frac{l_{\alpha_{j-1}\alpha_j}}{p_{\alpha_{j-1}\alpha_j}}, j = 1, 2, 3....$ We use the following notation for the partial sum:

$$\theta_i(h) = \frac{h_{\alpha_0}}{p_{\alpha_0}} \Sigma_{j=0}^{i} W_j b_{\alpha_j} \tag{5}$$

It is shown that $E(\theta_i(h)) = \langle h, \Sigma_{m=0}^{i} L^m b\rangle = \langle h, x^{(i+1)}\rangle$ and $E(\theta_i(h))$ tends to $\langle x, h\rangle$ as $i \to \infty$ [7]. To find r^{th} component of x, we put

$$h = (\underbrace{0, 0..., 1}_{r}, 0, ..., 0).$$

It follows that

$$\langle h, x\rangle = x_r.$$

The number of Markov chain is given by $N \geq (\frac{0.6745}{\epsilon} \frac{\|b\|}{(1-\|L\|)})^2$. With considering N paths ${\alpha_0}^{(m)} \to {\alpha_1}^{(m)} \to ... \to {\alpha_k}^{(m)}$, $m = 1, 2, 3..., N$, on the coefficient matrix, we have the Monte Carlo estimated solution by

$$\Theta_i(h) = \frac{1}{N} \Sigma_{m=1}^{N} \theta_i^{(m)}(h) \simeq \langle h, x^{(i+1)}\rangle.$$

The condition $\|L\| \leq 1$ is not very strong. In [9] and [10], it is shown that, it is possible to consider a Monte Carlo algorithm for which the Neumann series does not converge.

In this paper, we continue research on resolvent Monte Carlo algorithms presented in [4] and developed in [2], [3] and [5]. We consider Monte Carlo algorithms for solving linear systems in the case when the corresponding Neumann series does not necessarily converge. We apply a mapping of the spectral parameter λ to obtain a convergent algorithm. First, sufficient conditions for the convergence of the resolvent operator are given. Then the Monte Carlo algorithm is employed.

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2 Resolvent Operator Approach

2.1 The Convergence of Resolvent Operator

We study the behaviour of the equation

$$x - \lambda Lx = b$$

depending on the parameter λ. Define nonsingular values of L by

$$\pi(L) = \{\lambda |\ x - \lambda Lx = b \textit{ has a unique solution}\}.$$

$\chi(L) = (\pi(L))^c$ is called the characteristic set of L. Let X and Y be Banach spaces and let $U = \{x \in X : \|x\| \leq 1\}$. An operator $L : X \to Y$ is called compact if the closure of $L(U)$ is compact. By Theorem 4.18 in [8], if dimension the rang of L is finite, then L is compact. The statement that $\lambda \in \pi(L)$ is equivalent to asserting the existence of the two-sided inverse operator $(I - \lambda L)^{-1}$. It is shown that for compact operators, λ is a characteristic point of L if and only if $\frac{1}{\lambda}$ is an eigenvalue of L. Also, it is shown that for every $r > 0$, the disk $|\lambda| < r$ contains at most a finite number of characteristic values.

The operator R_λ defined by $R_\lambda = (I - \lambda L)^{-1}$ is called the resolvent of L, and

$$R_\lambda = I + L + \lambda L^2 + ... + \lambda^n L^{n+1} + ..$$

The radius of convergence r of the series is equal to the distance r_0 from the point $\lambda = 0$ to the characteristic set $\chi(L)$. Let $\lambda_1, \lambda_2, ..$ be the characteristic values of L that $|\lambda_1| \leq |\lambda_2| \leq ....$ The systematic error of the above presentation when m terms are used is

$$O((\frac{|\lambda|}{|\lambda_1|})^{m+1} m^{\rho-1})$$

where ρ is multiplicity of roots λ_1. This follows that when $|\lambda| \geq |\lambda_1|$ the series does not converge. In this case we apply the analytical method in functional analysis. The following theorem for the case of compact operators has been proved in [6].

Theorem 1. *Let λ_0 be a characteristic value of a compact operator L. Then, in a sufficiently small neighbourhood of λ_0, we have the expansion*

$$R_\lambda = ... + \frac{L_{-r}}{(\lambda - \lambda_0)^r} + ... + \frac{L_{-1}}{(\lambda - \lambda_0)} + L_0 + L_1(\lambda - \lambda_0) + ... + L_n(\lambda - \lambda_0)^n + ... \quad (6)$$

Here r is the rank of characteristic value λ_0, the operators $L_{-r}, ..., L_{-1}$ are finite dimensional and $L_{-r} \neq 0$. The series on the right-hand side of (6) is convergent in the space of operators $B(X, X)$.

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3 The Convergence of Monte Carlo Method

Let $\lambda_1, \lambda_2, ..$ be real characteristic values of L such that $\lambda_k \in (-\infty, -a]$. In this case we may apply a mapping of the spectral parameter λ. We consider a domain Ω lying inside the definition domain of R_λ as a function of λ such that all characteristic values are outside of Ω, $\lambda_* = 1 \in \Omega$, $0 \in \Omega$. Define $\psi(\alpha) = \frac{4a\alpha}{(1-\alpha)^2}$, $(|\alpha| < 1)$, which maps $\{\alpha : |\alpha| < 1\}$ to Ω described in [1]. Therefore the resolvent operator can be written in the form

$$R_\lambda b \simeq \Sigma_{k=1}^m b_k \alpha^k = \Sigma_{k=1}^m \Sigma_{i=1}^k d_i^{(k)} c_i \alpha^k$$
$$= \Sigma_{k=1}^m \Sigma_{j=k}^m d_k^{(j)} \alpha^j c_k = \Sigma_{k=1}^m g_k^{(m)} c_k$$

where $g_k^{(m)} = \Sigma_{j=k}^m d_k^{(j)} \alpha^j$ and $c_k = L^{k+1} b$. In [1], it is shown that $d_k^{(j)} = (4a)^k C_{k+j-1}^{2k-1}$. All in all, in the following theorem, it is shown that the random variable whose mathematical expectation is equal to $\langle h, \Sigma_{k=0}^m L^k \rangle$, is given by the following expression:

$$\Theta_m^*(h) = \frac{h_{\alpha_0}}{p_{\alpha_0}} \Sigma_{\nu=0}^m g_\nu^{(m)} W_\nu b_{\alpha_\nu}$$

where $W_0 = 1, W_j = W_{j-1} \frac{l_{\alpha_{j-1}\alpha_j}}{p_{\alpha_{j-1}\alpha_j}}, j = 1, 2, 3..., g_0^{(m)} = 1$ and $\alpha_0, \alpha_1, ...$ is a Markov chain with initial probability p_{α_0} and one step transition probability $p_{\alpha_{\nu-1}\alpha_\nu}$ for choosing the element $l_{\alpha_{\nu-1}\alpha_\nu}$ of the matrix L [1].

Theorem 2. *Consider matrix L, whose Neumann series does not necessarily converge. Let $\psi(\alpha) = \frac{4a\alpha}{(1-\alpha)^2}$ be the required mapping, so that the presentation $g_k^{(m)}$ exists. Then*

$$E\{\lim_{m\to\infty} \frac{h_{\alpha_0}}{p_{\alpha_0}} \Sigma_{\nu=0}^m g_\nu^{(m)} W_\nu b_{\alpha_\nu}\} = \langle h, x \rangle$$

In [5], authors have analysed the robustness of the Monte Carlo algorithm for solving a class of linear algebra problems based on bilinear form of matrix powers $\langle h, L^k b \rangle$. In [5], authors have considered real symmetric matrices with norms smaller than one. In this paper, results are extended considerably compared to cases [5] and [3]. We consider singular matrices. For matrices that are stochastic matrices the accuracy of the algorithm is particularly high.

3.1 Numerical Tests

In this section of paper we employed our resolvent Monte Carlo algorithm for solving systems of singular linear algebraic equations. The test matrices are randomly generated. The factor of the improvement of the convergence depends on parameter α . An illustration of this fact is Table 1. We consider randomly generated matrices of order 100, 1000 and 5000. But more precise consideration shows that the error decreases with the increasing of the matrix size.

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Table 1. Resolvent Monte Carlo results (number of trajectories $N = 10^5$)

Size n	Error
n=100	6.7668×10^{-4}
n=500	2.3957×10^{-5}
n=1000	1.1487×10^{-5}
n=2000	6.3536×10^{-6}
n=3000	3.8250×10^{-6}
n=4000	3.0871×10^{-6}
n=5000	2.2902×10^{-6}
n=6000	2.0030×10^{-6}

4 Conclusion

A new Monte Carlo algorithm for solving singular linear systems of equations is presented in this paper. The approach is based on the resolvent operator R_λ convergence. In fact we construct an algorithm based on the mapping of the spectral parameter λ. The approach is applied to systems with singular matrices. The initial results show that for such matrices a fairly high accuracy can be obtained.

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